

THE CLASSIFICATION OF 2-CONNECTED 7-MANIFOLDS

DIARMUID CROWLEY AND JOHANNES NORDSTRÖM

ABSTRACT. We present a classification theorem for closed smooth spin 2-connected 7-manifolds M . This builds on the almost-smooth classification from the first author's thesis. The main additional ingredient is a generalisation of the Eells–Kuiper invariant for any closed spin 7-manifold, regardless of whether the spin characteristic class $p_M \in H^4(M)$ is torsion. In addition we determine the inertia group of 2-connected M – equivalently the number of oriented smooth structures on the underlying topological manifold – in terms of p_M and the torsion linking form.

1. INTRODUCTION

Throughout this paper M will be a closed smooth spin 7-manifold and all homeomorphisms and diffeomorphisms are assumed to preserve spin structures, unless otherwise stated. We shall present a smooth classification of 2-connected spin 7-manifolds. This requires an generalisation of the Eells–Kuiper invariant of M , classically defined only when the spin characteristic class p_M is torsion.

1.1. The classification. To any closed smooth spin 7-manifold M we shall associate the following algebraic invariants:

- The integral cohomology group $H^4(M)$, which is a finitely generated abelian group.
- The spin characteristic class p_M , which is an even element of $H^4(M)$. It is a homeomorphism invariant (see Remark 2.1) which is related to the first Pontrjagin class by $2p_M = p_1(M)$.
- The torsion linking form $b_M : TH^4(M) \times TH^4(M) \rightarrow \mathbb{Q}/\mathbb{Z}$, which is a torsion form on the torsion subgroup $TH^4(M) \subseteq H^4(M)$ (by this we mean that b_M is symmetric, bilinear and nonsingular).
- The *quadratic linking family* q_M° , a family of quadratic refinements of $(H^4(M), b_M, p_M)$.
- The *generalised Eells–Kuiper invariant* μ_M , a mod 28 Gauss refinement of $(H^4(M), q_M^\circ, p_M)$.

For the moment we merely indicate the type of the last two invariants.

A quadratic refinement of b_M is a function $q : TH^4(M) \rightarrow \mathbb{Q}/\mathbb{Z}$ which satisfies the equation $q(x + y) = q(x) + q(y) + b_M(x, y)$, and we denote the set of such q by $\mathcal{Q}(b_M)$. The homogeneity defect of $q \in \mathcal{Q}(b_M)$ is the unique element $\beta \in 2TH^4(M)$ such that $q(x) - q(-x) = b_M(x, \beta)$. Let

$$S_2 := \{h \in H^4(M) \mid p_M - 2h \text{ is torsion}\}.$$

That q_M° is a *family of quadratic refinements* of $(H^4(M), b_M, p_M)$ means that it is a function

$$q_M^\circ : S_2 \rightarrow \mathcal{Q}(b_M), \quad h \mapsto q_M^h,$$

such that q_M^h has homogeneity defect $\beta_h := p_M - 2h$, and satisfies a transformation rule (see Definition 2.13). Let d_π be the greatest integer dividing p_M modulo torsion (or $d_\pi := 0$ if p_M is a torsion element), $\tilde{d}_\pi := \text{lcm}(4, d_\pi)$ and $\hat{d}_\pi := \gcd(\frac{\tilde{d}_\pi}{4}, 28)$. Let

$$S_{d_\pi} := \{k \in H^4(M) \mid p_M - d_\pi k \text{ is torsion}\}$$

and set $\beta_k := p_M - d_\pi k$ for each $k \in S_{d_\pi}$. By saying that the generalised Eells–Kuiper invariant of M is a *mod 28 Gauss refinement* of $(H^4(M), q_M^\circ, p_M)$ we mean that it is a function

$$\mu_M : S_{d_\pi} \rightarrow \mathbb{Q}/\hat{d}_\pi \mathbb{Z},$$

such that $\mu_M(k) = A(q^{\frac{d_\pi}{2}k}) \bmod \mathbb{Z}$, (where A is the Arf invariant of a quadratic refinement, computed in terms of a Gauss sum in (8)) and such the transformation rule

$$\mu_M(k+t) - \mu_M(k) = \frac{d_\pi^2 b_M(t, t) - 2d_\pi b_M(\beta_k, t)}{8} \bmod \gcd(28, \frac{d_\pi}{4}), \quad (1)$$

holds for all $k \in S_{d_\pi}$ and $t \in T$. For the purposes of this introduction, the main significance of these conditions is that a Gauss refinement is defined by its value at a single element in S_{d_π} , and that the difference between two Gauss refinements of $(H^4(M), q_M^\circ, p_M)$ is just a constant in $\mathbb{Z}/\hat{d}_\pi \mathbb{Z}$.

Remark 1.1. If p_M is a torsion element then $d_\pi = 0$ and $\hat{d}_\pi = 28$, while $S_{d_\pi} = T$ contains the distinguished element 0. The value $\frac{1}{28}\mu_M(0) \in \mathbb{Q}/\mathbb{Z}$ recovers the original Eells-Kuiper invariant.

If G is a finitely generated abelian group, $p \in 2G$ and b is a torsion form on T then we call (G, b, p) a *base*. If q is a family of quadratic refinements of (G, b, p) then we call (G, q, p) a *refinement*; we suppress b since it can be recovered from q . If μ is a mod 28 Gauss refinement of (G, q°, p) then we call the quadruple (G, q°, μ, p) a *mod 28 distillation*. If $F: G' \rightarrow G$ is a group isomorphism then we can define another mod 28 distillation $(G', F^\#q, F^\#\mu, F^\#p)$ by pulling back: $F^\#(p) := F^{-1}(p)$, $(F^\#q)^h(x) := q^{F(h)}(F(x))$, and $F^\#\mu := \mu \circ F$.

The mod 28 distillation $(H^4(M), q_M^\circ, \mu_M, p_M)$ of M is an invariant of spin diffeomorphisms: if $f: M \rightarrow M'$ is a spin diffeomorphism then $f^*: H^4(M') \rightarrow H^4(M)$ is an isomorphism, and $(q_{M'}, \mu_{M'}, p_{M'}) = ((f^*)^\#q, (f^*)^\#\mu, (f^*)^\#p)$. In fact, only μ_M depends on the smooth structure, and the refinement $(H^4(M), q_M^\circ, p_M)$ is also invariant under spin almost diffeomorphisms.

An almost diffeomorphism $f: M_0 \cong M_1$ is a homeomorphism which is smooth except perhaps at a finite number of points. It follows from results of the first author's thesis, see Lemma 3.2, that 2-connected 7-manifolds are classified up to homeomorphism and almost diffeomorphism by their refinements.

Theorem 1.2. *Every refinement (G, q°, p) is isomorphic to $(H^4(M), q_M^\circ, p_M)$ for some 2-connected M . Moreover, if M_0 and M_1 are 2-connected, then an isomorphism $F: H^4(M_1) \rightarrow H^4(M_0)$, is realised by an almost diffeomorphism $f: M_0 \cong M_1$ such that $F = f^*$ if and only if $(q_{M_1}^\circ, p_{M_1}) = F^\#(q_{M_0}^\circ, p_{M_0})$.*

The same statement holds when “almost diffeomorphism” is replaced by “homeomorphism”.

In this paper we prove that the generalised Eells-Kuiper invariant is precisely what needs to be added to Theorem 1.2 to obtain a smooth classification of 2-connected 7-manifolds.

Theorem 1.3. *Every mod 28 distillation (G, q°, μ, p) is isomorphic to $(H^4(M), q_M^\circ, \mu_M, p_M)$ for some 2-connected M . Moreover, for any pair of 2-connected 7-manifolds M_0 and M_1 , an isomorphism $F: H^4(M_1) \rightarrow H^4(M_0)$, is realised by a diffeomorphism $f: M_0 \cong M_1$ such that $F = f^*$ if and only if $(q_{M_1}^\circ, \mu_{M_1}, p_{M_1}) = F^\#(q_{M_0}^\circ, \mu_{M_0}, p_{M_0})$.*

1.2. Elaboration of the classification. If we are simply interested in whether M_0 and M_1 are diffeomorphic, we can consider the following simpler invariant. Let $TH^4(M) \subset H^4(M)$ be the torsion subgroup, let $\text{Aut}(b_M)$ be the group of automorphisms of the linking form b_M and recall that $\beta_k = p_M - d_\pi k \in T$ for each $k \in S_{d_\pi}$. We define the *reduced splitting function* of M (the terminology is explained in Section 3.5) to be the function

$$\overline{q}_M^\circ: S_{d_\pi} \rightarrow (2TH^4(M)/\text{Aut}(b)) \times \mathbb{Q}/\hat{d}_\pi \mathbb{Z}, \quad k \mapsto ([\beta_k], \mu(k)).$$

Theorem 1.4. *Let M_0 and M_1 be 2-connected with $(H^4(M_0), b_{M_0}, p_{M_0}) \cong (H^4(M_1), b_{M_0}, p_{M_1})$ and reduced smooth splitting functions \overline{q}_0° and \overline{q}_1° . The following are equivalent:*

- (i) M_0 is diffeomorphic to M_1 ;
- (ii) $\overline{q}_0^\circ(S_{d_\pi}) = \overline{q}_1^\circ(S_{d_\pi})$;
- (iii) $\overline{q}_0^\circ(S_{d_\pi}) \cap \overline{q}_1^\circ(S_{d_\pi}) \neq \emptyset$.

The corresponding result for almost diffeomorphisms is given in Corollary 3.6.

We can also reformulate the above classification in categorical language as follows. Let \mathcal{Q}_μ denote the category of mod 28 distillations (G, q°, μ, p) with morphisms isomorphisms:

$$\text{Ob}(\mathcal{Q}_\mu) = \{(G, q^\circ, \mu, p)\}.$$

Let $\mathcal{M}_{7,2}^{spin}$ denote the category of 2-connected spin 7-manifolds with morphisms diffeomorphisms:

$$\text{Ob}(\mathcal{M}_{7,2}^{spin}) = \{M \mid \pi_1(M) = 0 = \pi_2(M)\}.$$

Given a diffeomorphism $f: M_0 \cong M_1$, write $f^*: H^4(M_1) \cong H^4(M_0)$ for the induced action on cohomology. Hence we obtain the contravariant functor

$$\mathcal{Q}: \mathcal{M}_{7,2}^{spin} \rightarrow \mathcal{Q}_\mu, \quad \begin{cases} M & \mapsto (H^4(M), q_M^\circ, \mu_M, p_M), \\ f: M_0 \cong M_1 & \mapsto f^*. \end{cases}$$

The operations of connected sum and reversing orientation in $\mathcal{M}_{7,2}^{spin}$ are mirrored by corresponding operations in \mathcal{Q}_μ . The orthogonal sum of two distillations $(G_i, q_i^\circ, \mu_i, p_i)$, $i = 0, 1$, is defined by

$$(G_0, q_0^\circ, \mu_0, p_0) \oplus (G_1, q_1^\circ, \mu_1, p_1) := (G_0 \oplus G_1, q_0^\circ \oplus q_1^\circ, \mu_0 \oplus \mu_1, p_0 \oplus p_1).$$

For the refinements, we note that $S_2(G_0 \oplus G_1) = S_2(G_0) \times S_2(G_1)$, and for refinements $q_i^{h_i}$ of b_i , $q_0^{h_0} \oplus q_1^{h_1}$ refines $b_0 \oplus b_1$. For the mod 28 distillations, we have that $c_i d_{\pi_i} = d_{\pi_0 \oplus \pi_1}$ for some integer c_i . In this case $c_0 S_{d_{\pi_0}} \times c_1 S_{d_{\pi_1}} \subseteq S_{d_{\pi_0 \oplus \pi_1}}$ and we set

$$(\mu_0 \oplus \mu_1)(c_0 k_0 + c_1 k_1) = \mu_0(k_0) + \mu_1(k_1) \pmod{\gcd\left(28, \frac{d_{\pi_0 \oplus \pi_1}}{4}\right)}.$$

Since $\mu_0 \oplus \mu_1$ is determined by its value on a single $k \in S_{d_{\pi_0 \oplus \pi_1}}$, this suffices to define the sum of distillations. We define the negative of a distillation by

$$-(G, q^\circ, \mu, p) = (G, -q^\circ, -\mu, p).$$

Theorem 1.5. *The functor $\mathcal{Q}: \mathcal{M}_{7,2}^{spin} \rightarrow \mathcal{Q}_\mu$ is surjective and faithful. Moreover*

- (i) $\mathcal{Q}(M_0 \# M_1) = \mathcal{Q}(M_0) \oplus \mathcal{Q}(M_1)$,
- (ii) $\mathcal{Q}(-M) = -\mathcal{Q}(M)$.

Remark 1.6. A homotopy classification of 2-connected M is given in [5, Theorem 6.11].

1.3. The generalised Eells-Kuiper invariant. Since the smooth classification of 2-connected rational homology spheres and the classification of general 2-connected M up to connected sum with homotopy spheres was already given in [5] (with some cosmetic differences in the description of the quadratic linking families, *cf.* Remark 2.19), the novelty of Theorem 1.5 lies in the smooth classification when $H^4(M)$ is infinite. The key ingredient is the generalisation of the Eells-Kuiper invariant.

Let X be a closed spin 8-manifold. By the index theorem [2, Theorem 5.3] $\hat{A}(X)$, the \hat{A} -genus of X , is equal to the index of the Dirac operator on X , and so an integer. The classical Eells-Kuiper invariant is derived from on the relation

$$p_X^2 - \sigma(X) = 224\hat{A}(X), \tag{2}$$

where X has signature $\sigma(X)$ and spin characteristic class p_X : the latter is defined in Section 2.1. By the index theorem $\hat{A}(X)$ is an integer. If M is a closed 7-manifold such that p_M is a *torsion* class (so rationally trivial) and W is a spin coboundary of M , then p_W^2 has a well-defined integral over W (it might in general take values in \mathbb{Q} and not just \mathbb{Z}), and (2) implies that

$$\mu(M) := \frac{p_W^2 - \sigma(W)}{8} \in \mathbb{Q}/28\mathbb{Z} \tag{3}$$

is independent of the choice of coboundary W . This defines the classical Eells-Kuiper invariant, modulo normalisation by a factor of 28.

To define an analogue when p_M is not a torsion class we have to let it take values not modulo 28 but modulo the integer $\hat{d}_\pi = \gcd(\frac{\hat{d}_\pi}{4}, 28)$, depending on the divisibility of p_M modulo torsion as above. Moreover, the generalisation is not simply a constant in $\mathbb{Q}/\hat{d}_\pi\mathbb{Z}$ but a function.

Now suppose that W is a spin coboundary of M , and that there exists $n \in H^4(W)$ such that the image of $p_W - d_\pi n$ under the restriction map $H^4(W) \rightarrow H^4(M)$ is a torsion class; equivalently

$jn \in S_{d_\pi}$. (If W is 3-connected then such n exist, and any spin M has 3-connected coboundaries: see the start of Section 2.2). We define

$$g_W(jn) := \frac{(p_W - d_\pi n)^2 - \sigma(W)}{8} \in \mathbb{Q}/\frac{\hat{d}_\pi}{4}\mathbb{Z}. \quad (4)$$

and extend g_W to a function $S_{d_\pi} \rightarrow \mathbb{Q}/\frac{\hat{d}_\pi}{4}\mathbb{Z}$ by the transformation rule (1). Then g_W is independent of the choices of n . The following lemma (cf. (24)) implies that $\mu_M := g_W \bmod \hat{d}_\pi$ is independent of the choice of W , and functorial.

Lemma 1.7. *Let $f : \partial W_0 \rightarrow \partial W_1$ be a spin diffeomorphism and $X := (-W_0) \cup_f W_1$. Then*

$$g_{W_1} - (f^*)^\# g_{W_0} = 28\hat{A}(X) \bmod \frac{\hat{d}_\pi}{4}.$$

The idea of the definition is that the simplest way to change (3) to something that is well-defined when the restriction of p_W to the boundary is rationally non-trivial is to compensate by subtracting from p_W a class that is divisible by d_π and has the same rational image in $H^4(M)$. The essentially different ways of doing that are parametrised by S_{d_π} , and that is why we end up with the generalised Eells-Kuiper invariant, $\mu_M : S_{d_\pi} \rightarrow \mathbb{Q}/\hat{d}_\pi\mathbb{Z}$, being a function defined on S_{d_π} .

The definition of the s_1 invariant by Kreck and Stolz [24] provides as a byproduct a way to compute the classical Eells-Kuiper invariant in terms of coboundaries that are not spin, but merely spin^c . Proposition 2.33 gives a similar way to compute the generalised Eells-Kuiper invariant via spin^c coboundaries. This turns out to be useful in an application to distinguishing between closed 7-manifolds with holonomy G_2 that are homeomorphic but not diffeomorphic.

1.4. Inertia and reactivity. Let $\Theta_7 = \{\Sigma \mid \Sigma \simeq S^7\}$ be the group of spin diffeomorphism classes of homotopy 7-spheres Σ . Since homotopy spheres are simply connected, this is equivalent to the standard definition of Θ_7 in [22]. By [22], Θ_7 is an abelian group under connected sum and $\Theta_7 \cong \mathbb{Z}/28$. We define the *inertia group* of M to be the following subgroup of Θ_7 :

$$I(M) := \{\Sigma \mid M \# \Sigma \cong M\}.$$

Remark 1.8. Let M_+ denote the oriented manifold underlying M . If M is simply connected then $I(M) = I(M_+)$, where $I(M_+)$ is the usual inertia group of M_+ , which is defined using orientation preserving diffeomorphisms $f_+ : M_+ \# \Sigma_+ \cong M_+$.

It turns out that even with Theorem 1.5 in hand, the determination of $I(M)$ can be a delicate problem. The reason is that μ_M is not a constant but rather a function and so it is possible for almost diffeomorphisms of M to act non-trivially on μ_M . Equivalently, the automorphism group of a refinement (G, q°, p) can act non-trivially on the set of mod 28 Gauss refinements.

The inertia group is closely related to what we (therefore) call the *reactivity* of M . Let $\text{ADiff}(M)$ denote the group of spin almost diffeomorphisms of M . Given $f \in \text{ADiff}(M)$, the mapping torus T_f of f has as well-defined spin characteristic class $p_{T_f} \in H^4(T_f)$ and we define the integer $p^2(f) := \langle p_{T_f}^2, [T_f] \rangle$. This defines a homomorphism

$$p^2 : \text{ADiff}(M) \rightarrow \mathbb{Z}, \quad f \mapsto p^2(f),$$

and the reactivity of M is the non-negative integer $R(M)$ defined by

$$p^2(\text{ADiff}(M)) = R(M)\mathbb{Z}. \quad (5)$$

Clearly $R(M)$ is an almost diffeomorphism invariant of M . Since T_f has zero signature and p_{T_f} is characteristic for the intersection form of T_f we have $R(M) \in 8\mathbb{Z}$. It is well understood that $f \in \text{ADiff}(M)$ is pseudo-isotopic to a diffeomorphism if and only if $p^2(f)$ is divisible by 224 (see Lemma 3.7) and consequently

$$I(M) = \frac{R(M)}{8} \Theta_7. \quad (6)$$

To determine $R(M)$ when M is 2-connected, we first determine values of $p^2(f)$ that are realised for $H^*(f) = \text{Id}$ (see Proposition 3.10). This reduces the determination of $R(M)$ to understanding the action of the automorphism group $\text{Aut}_{q^\circ}(H^4(M))$ of $(H^4(M), q_M^\circ, p_M)$ on mod 28 Gauss refinements. That can in turn be reduced to understanding the automorphism group $\text{Aut}_b(H^4(M))$

of $(H^4(M), b_M, p_M)$, which is much easier to deal with in practice. In fact, $R(M)$ is almost completely determined just using the following ‘intermediate’ notion of the divisibility of p_M , whose significance was pointed out by Wilkens [39, Conjecture p. 548]:

$$d_o := \begin{cases} 0 & \text{if } p_M \text{ is torsion,} \\ \text{Max}\{s \mid s, m \in \mathbb{Z}, sm^2 \text{ divides } mp_M\} & \text{otherwise.} \end{cases}$$

Corollary 4.17 and (6) give the next theorem, where for a fraction $\frac{a}{b}$ in lowest terms $\text{Num}(\frac{a}{b}) = a$.

Theorem 1.9. *Let M be 2-connected and let $d_o = d_o(M)$. There is an integer $r \in \{0, 1, 2\}$ depending only on the base $(H^4(M), b_M, p_M)$, such that*

$$R(M) = \text{lcm}(8, 2^r d_o).$$

Consequently

$$I(M) = \text{Num}\left(\frac{2^r d_o}{8}\right) \Theta_7.$$

If $TH^4(M)$ has odd order then $r = 1$.

If $H^4(M)$ does have some 2-torsion then in general one needs to look at the torsion linking form in detail to determine r . Wilkens’ conjecture [39, Conjecture p. 548] for the inertia group is equivalent to supposing that $r = 1$ always, which is not true. The invariant $r = r(G, b, p)$ is defined in Definition 4.5 and while we do not have a closed formula for r , it is feasible to compute r for any given example. For examples where $r = 0, 1$ or 2 , see Example 5.2.

We next discuss some consequences of Theorem 1.9 and its proof. If N is a close smooth manifold, let $n_+(N)$ denote the number of oriented diffeomorphism classes of smooth structures on the topological manifold underlying M . From Theorems 1.2 and 1.9 and Remark 1.8 we deduce

Corollary 1.10. *If M is 2-connected then $n_+(M) = \text{gcd}(\text{Num}(2^{r-3}d_o), 28)$.*

We call a homotopy equivalence $f: N_0 \simeq N_1$ of smooth manifolds if there is a stable vector bundle isomorphism $f^* \tau_{N_1} \cong \tau_{N_0}$ where τ_{N_i} is the stable tangent bundle of N_i . In Lemma 5.5 we show that a homotopy equivalence $f: M_0 \simeq M_1$ of 2-connected 7-manifolds with $(f^*)^\sharp p_{M_1} = p_{M_0}$ is tangential. Together with Theorem 1.9 this entails

Corollary 1.11. *Let M_0 and M_1 be 2-connected and let $f: M_0 \simeq M_1$ be a tangential homotopy equivalence. Then $I(M_0) = I(M_1)$.*

One may wonder if Corollary 1.11 is true because tangentially homotopy equivalent 2-connected 7-manifolds are almost diffeomorphic (equivalently homeomorphic by Theorem 1.2). However this was shown not to be the case in [5, p. 114], contradicting results of Madsen, Taylor and Williams [26, Theorem C and Theorem 5.10]: see Proposition 5.6 and Remark 5.7.

The computation of the reactivity of M also has applications in G_2 -topology. We define the *smooth reactivity* of M , $R^{\text{Diff}}(M)$, using the equation $p^2(\text{Diff}(M)) = R^{\text{Diff}}(M)\mathbb{Z}$. By Corollary 4.17,

$$R^{\text{Diff}}(M) = \text{lcm}(2^r d_o, 224),$$

and in [6, Section 6] we show that $R^{\text{Diff}}(M)$ determines the number of G_2 -structures on M modulo homotopies and diffeomorphisms.

The proof of Theorem 1.9 leads to information about the mapping class groups of M . Let $I_H(M) \subseteq I(M)$ be the subgroup of the inertia group of M consisting of homotopy spheres Σ such that there is a diffeomorphism $f: M \# \Sigma \cong M$ where $H^*(f) = \text{Id}$, considering $M \# \Sigma$ and M as the same topological space. Using the delicate algebra in Section 4.3 we construct a surjective homomorphism

$$\hat{P}: \text{Aut}_{q^\circ}(H^4(M)) \rightarrow I(M)/I_H(M),$$

such that $F \in \text{Aut}_{q^\circ}(H^4(M))$ is realised by a diffeomorphism of M if and only if $\hat{P}(F) = 0$. Now by Theorem 1.2, every $F \in \text{Aut}_{q^\circ}(H^4(M))$ is realised by an almost diffeomorphism of M and in Proposition 6.4 we prove that every subquotient of Θ_7 can be realised as the pair $I(M)/I_H(M)$ for some 2-connected M . As consequence we have

Theorem 1.12. *There exist 2-connected M with automorphisms $F \in \text{Aut}_{q^o}(H^4(M))$ which are not realised by any diffeomorphism of M . Necessarily every such F is realised by an almost diffeomorphism of M .*

1.5. Organisation. The rest of this paper is organised as follows. In Section 2 we define the invariants used in Theorems 1.2 and 1.3. In particular, families of refinements, Gauss refinements and the generalised Eells-Kuiper invariant are defined in Sections 2.4, 2.5 and 2.6 respectively. In Section 3 we prove our main classification results and we discuss the connected sum splitting of 2-connected 7-manifolds in Theorems 3.5 and 3.14. Section 4 is an algebraic section in which we analyse the automorphisms of refinements and bases and the action of these automorphisms on Gauss refinements. This section contains the proof of Theorem 1.9 which follows from the computation of the reactivity of M in Corollary 4.17.

In Section 5 we illustrate our classification of 2-connected M with examples: we consider the total spaces of 3-sphere bundles over S^4 , examples admitting rare metrics and we give a pair of manifolds M_0 and M_1 which are tangentially homotopy equivalent but not homeomorphic. We also we present a refinement of Wilkes' identification of the set of indecomposable generators for the monoid of almost diffeomorphism classes of 2-connected 7-manifolds under the operation of connected sum: see Theorem 5.8. In Section 6 we investigate the relationship between the inertia groups of M and the mapping class groups of M and prove Theorem 1.12.

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2. INVARIANTS

In this section we define the invariants needed to classify 2-connected spin 7-manifolds M . In Section 2.1 we introduce the linking form of b_M of M and the spin characteristic class $p_M \in 2H^4(M)$.

In Section 2.2 we recall that every M has a 3-connected spin coboundary W which has an intersection form

$$\lambda_W: FH^4(W, \partial W) \times FH^4(W, \partial W) \rightarrow \mathbb{Z},$$

where $H^4(W, \partial W) = H^4(W, \partial W)$ and $FH^4(W, \partial W)$ is the free quotient of $H^4(W, \partial W)$. A key fact is that p_W is characteristic for λ_W , i.e. $\lambda_W(x, x) = x \cup p_W \bmod 2$ for all $x \in H^4(W, \partial W)$. We call a triple $(H^4(W, \partial W), \lambda_W, p_W)$ a *characteristic form* and identify it as the salient algebraic model for W .

In sections Sections 2.3, 2.4 and 2.5 we progressively build algebraic “boundary invariants” of characteristic forms. Section 2.3 recalls the theory of refinements of torsion forms and Section 2.4 shows how a characteristic form defines a family of refinements on its boundary. In Section 2.5 we define the generalised Eells-Kuiper invariant μ_M of M using the characteristic form $(FH^4(W, \partial W), \lambda_W, p_W)$ of a spin coboundary W and Hirzebruch's characteristic class formulae for the \hat{A} -genus and the L -genus. The generalised Eells-Kuiper invariant is a reduced defect invariant of the \hat{A} -genus. Finally in Section 2.7 we show how μ_M can be computed via a coboundary W which is spin^c rather than spin .

2.1. Basic invariants. To any closed spin 7-manifold we can associate its integral cohomology group $H^4(M)$ and torsion linking form

$$b_M: TH^4(M) \times TH^4(M) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

defined on the torsion subgroup of $H^4(M)$. The linking form b_M is a nonsingular symmetric bilinear form. We also have the spin characteristic class $p_M \in 2H^4(M)$ whose definition we recall below.

The classifying space $BSpin$ is 3-connected and $\pi_4(BSpin) \cong \mathbb{Z}$. It follows that $H^4(BSpin) \cong \mathbb{Z}$ is infinite cyclic. A generator is denoted $\pm \frac{p_1}{2}$ and the notation is justified since for the canonical

map $\pi: BSpin \rightarrow BSO$ we have $\pi^*p_1 = 2\frac{p_1}{2}$ where p_1 is the first Pontrjagin class. Given a spin manifold N we write

$$p_N := \frac{p_1}{2}(N) \in H^4(N).$$

The triple $(H^4(M), b_M, p_M)$ is called the base of M and for later use we introduce the category \mathcal{Q}_b consisting of bases with morphisms isomorphisms

$$\text{Ob}(\mathcal{Q}_b) = \{(G, b, p)\}.$$

Remark 2.1. In order to prove the topological invariance of invariants we define in the later subsections, we consider p_Y for general topological spin manifolds Y . We let $BTop$ denote the classifying space for stable topological microbundles, see [23, Essay IV, Proposition 8.1], and $BTop\langle 4 \rangle$ its 3-connected cover. Equivalently, $BTop\langle 4 \rangle$ is the classifying space for stable spin topological microbundles. By [18, (3)] there is a split short exact sequence

$$0 \rightarrow \pi_4(BSpin) \rightarrow \pi_4(BTop\langle 4 \rangle) \rightarrow \mathbb{Z}/2 \rightarrow 0.$$

It follows that the canonical homomorphism $H^4(BTop\langle 4 \rangle) \rightarrow H^4(BSpin)$ is an isomorphism and so $p_N \in H^4(N)$ is defined for any topological spin manifold.

By [24, Lemma 6.5], the mod 2 reduction of \pm is the 4th Stiefel-Whitney class w_4 . That has the following consequences for the parity of the characteristic class of a closed spin manifold.

Lemma 2.2.

- (i) Let M be a closed spin 7-manifold. Then $p_M \in 2H^4(M)$.
- (ii) Let X be a closed spin 8-manifold. For all $x \in H^4(X; \mathbb{Z}/2)$

$$x^2 = x \cup p_Y \in H^8(X; \mathbb{Z}/2).$$

- (iii) Let W be a compact spin 8-manifold with boundary M . For all $x \in H^4(W, M; \mathbb{Z}/2)$

$$x^2 = x \cup p_W \in H^8(W, M; \mathbb{Z}/2).$$

Proof. By Wu's formula, see *e.g.* [31, Theorem 11.14], $w_4 = v_4$ for any closed spin manifold since the first three Wu classes of a spin manifold vanish.

- (i) Now $v_4(M) = 0$ since M is 7-dimensional, the Wu class satisfies $v_4(M) \cup x = Sq^4(x)$ for all $x \in H^3(M; \mathbb{Z}_2)$, and Sq^4 vanishes on degree three classes.
- (ii) $x^2 = Sq^4(x) = x \cup v_4(X) = x \cup p_X$.
- (iii) Let $X := W \cup_{\text{Id}_M} (-W)$. The push-forward $i_* : H^*(W, M) \rightarrow H^*(X)$ of the inclusion $i : W \hookrightarrow X$ is dual under the Poincaré pairing to the restriction $i^* : H^4(X) \rightarrow H^4(W)$. Since $i^*p_X = p_W$, (ii) gives

$$x^2 = (i_*x)^2 = i_*x \cup p_X = x \cup p_Y,$$

where the equalities take place in $\mathbb{Z}/2 \cong H^8(X; \mathbb{Z}/2) \cong H^8(W, M; \mathbb{Z}/2)$. \square

2.2. Algebraic models of coboundaries. Let M be a closed spin 7-manifold. Since the bordism group Ω_7^{Spin} vanishes by [29], there is a compact spin 8-manifold W such that $\partial W = M$. Applying surgery below the middle dimension to W [28, Theorem 3], we can assume that W is 3-connected. Since W is 3-connected the relative cohomology sequence of (W, M) gives exactness of

$$FH^4(W, \partial W) \rightarrow H^4(W) \rightarrow H^4(M) \rightarrow 0 \quad (7)$$

where $H^4(W)$ is torsion-free and $FH^4(W, \partial W) := H^4(W, \partial W)/\text{Tors}(H^4(W, \partial W))$ is the free quotient of $H^4(W, \partial W)$ (by Poincaré duality, the torsion subgroup of $H^4(W, \partial W)$ is isomorphic to the torsion subgroup of $H^3(M)$.) The first homomorphism in (7) can be thought of as the adjoint homomorphism $\hat{\lambda}_W : FH^4(W, \partial W) \rightarrow FH^4(W, \partial W)^*$ of the intersection pairing λ_W on $FH^4(W, \partial W)$. The principle is to regard $(FH^4(W, \partial W), \lambda_W)$ as a “model” for a coboundary W .

Let us set up some terminology to deal with these models. We say that (H, λ) is an integral form if H is a finitely generated free abelian group and $\lambda : H \times H \rightarrow \mathbb{Z}$ is symmetric and bilinear. Let $\hat{\lambda}$ denote the adjoint map $H \rightarrow H^*$. The “boundary” of (H, λ) is $G := \text{coker } \hat{\lambda}$. We say that an element $\alpha \in H^*$ is *characteristic* for λ if $\lambda(x, x) = \alpha(x) \bmod 2$ for all $x \in H$. We then call (H, λ, α) a *characteristic form*.

If W is a 3-connected coboundary of M then what we have said so far is that $(FH^4(W, \partial W), \lambda_W)$ is an integral form with boundary $H^4(M)$. By Lemma 2.2(iii), $(FH^4(W, \partial W), \lambda_W, p_W)$ is characteristic. (If M is 2-connected then Wall's classification of 3-connected 8-manifolds [34] ensures that the characteristic form of W is a complete invariant of diffeomorphisms: see [5, Corollary 2.5].) In the next subsections we study the structures that an integral or characteristic form induces on its boundary. By applying this to the algebraic model of a 3-connected coboundary of M we obtain the desired algebraic invariants of M . To prove that they are independent of the choice of W we will combine a splitting result for the algebraic constructions with the following lemma whose proof is a simple application of the Mayer-Vietoris theorem.

Lemma 2.3. *Let W_i be compact 3-connected spin 8-manifolds, $f : \partial W_0 \rightarrow \partial W_1$ a homeomorphism, and $X := (-W_0) \cup_f W_1$ (a closed spin topological manifold). Then $H^4(W_i, \partial W_i) \hookrightarrow H^4(X)$, and the images are orthogonal to each other with respect to the intersection form λ_X of X . Further the restriction map $H^4(X) \rightarrow H^4(M)$ is surjective, with kernel $H^4(W_0, \partial W_0) + H^4(W_1, \partial W_1)$. \square*

Remark 2.4. We note that by Lemma 2.2 (ii), the triple $(H^4(X), \lambda_X, p_X)$ of the manifold X in Lemma 2.3 is a (non-singular) characteristic form. Moreover, the image of p_X under the restriction map $H^4(X) \rightarrow H^4(W_i)$ is of course p_{W_i} .

2.3. Torsion forms and quadratic refinements on finite groups. Throughout this subsection T will be a *finite* abelian group. We say that $b : T \times T \rightarrow \mathbb{Q}/\mathbb{Z}$ is a *torsion form* on T if it is symmetric, bilinear, and nonsingular in the sense that the induced map $T \rightarrow \text{Hom}(T, \mathbb{Q}/\mathbb{Z})$ is an isomorphism. We call a function $q : T \rightarrow \mathbb{Q}/\mathbb{Z}$ a *quadratic refinement* of b if

$$q(x + y) = q(x) + q(y) + b(x, y), \quad \forall x, y \in T.$$

The *homogeneity defect* of q is the element $\beta \in 2T$ such that $q(x) - q(-x) = b(x, \beta)$ for all $x \in G$. If $\beta = 0$ then $q(x) = q(-x)$ and q is called *homogeneous*. We define

$$\mathcal{Q}(b) := \{q \mid q \text{ is quadratic refinement of } b\},$$

and we let $\mathcal{Q}^0(b) \subset \mathcal{Q}(b)$ be the set of homogeneous quadratic refinements of b . In this subsection we consider the problem of classifying the refinements in $\mathcal{Q}(b)$ up to isomorphism. For $\mathcal{Q}^0(b)$ this problem was solved by Nikulin [32] and the general solution was given independently by the first author [5] and Deloup and Massuyeau [10].

The first basic result [5, Lemma 2.30, Lemma 2.31] is that $\mathcal{Q}(b)$ and $\mathcal{Q}^0(b)$ are both non-empty and that G acts freely and transitively on $\mathcal{Q}(b)$ via the action

$$\mathcal{Q}(b) \times G \rightarrow \mathcal{Q}(b), \quad (q, a) \mapsto q_a,$$

where for all $x \in G$,

$$q_a(x) := q(x) + b(x, a) = q(x + a) - q(a).$$

Example 2.5. If $T \cong \mathbb{Z}/r\mathbb{Z}$ is cyclic then all torsion forms and refinements on T are given by the following examples. Given $\theta \in \mathbb{Z}/r$ coprime to r , let $\langle \frac{\theta}{r} \rangle$ denote T equipped with the torsion form

$$b(x, y) := \frac{\theta xy}{r} \in \mathbb{Q}/\mathbb{Z}.$$

Given $\theta \in \mathbb{Z}/2r$ coprime to r and $\gamma \in \mathbb{Z}/r$ (so that $2\gamma \in \mathbb{Z}/2r$), we define a quadratic refinement $\langle\langle \frac{\theta}{2r} \rangle\rangle_\gamma$ of $\langle \frac{\theta}{r} \rangle$ by

$$q(x) := \theta \left(\frac{x^2 + 2\gamma x}{2r} \right) \in \mathbb{Q}/\mathbb{Z}.$$

Beyond the homogeneity defect, we introduce two further equivalent invariants of q . The first of these is the *Gauss sum* of q which is the complex number

$$GS(q) := \sum_{x \in T} e^{2\pi i q(x)} \in \mathbb{C},$$

where $i = \sqrt{-1}$ and e is Euler's number. From the fact that $q_a(x) = q(x + a) - q(a)$ one easily obtains the following useful

Lemma 2.6 ([10, (4.1)]). $GS(q_a) = e^{-2\pi i q(a)} GS(q)$. \square

It is a theorem of Milgram [30, Theorem, p. 127, Appendix 4] that if q is homogeneous, then $GS(q)$ is a non-zero complex number with modulus $\sqrt{|T|}$: by Lemma 2.6, this holds for all $q \in \mathcal{Q}(b)$. We define the *Arf invariant* of q to be the number $A(q) \in \mathbb{Q}/\mathbb{Z}$ which is the argument of $GS(q)$ divided by 2π . That is

$$GS(q) = \sqrt{|T|} e^{2\pi i A(q)} \in \mathbb{C}. \quad (8)$$

Then Lemma 2.6 is equivalent to

$$A(q_a) = A(q) - q(a). \quad (9)$$

Before giving the classification theorems for $\mathcal{Q}(b)$, we review how elements of $\mathcal{Q}(b)$ can be presented as the boundaries of nondegenerate characteristic forms (H, λ, α) and how $A(q)$ is determined by (H, λ, α) in this situation. If λ is nondegenerate then the boundary $T := \text{coker}(\hat{\lambda})$ of (H, λ) fits into the short exact sequence

$$0 \rightarrow H \xrightarrow{\hat{\lambda}} H^* \xrightarrow{j} T \rightarrow 0.$$

It follows that $\hat{\lambda}$ becomes an isomorphism when tensored with the rationals, $\hat{\lambda}_{\mathbb{Q}}: H \otimes \mathbb{Q} \cong H^* \otimes \mathbb{Q}$. We can use the inverse $(\hat{\lambda}_{\mathbb{Q}})^{-1}$ to pullback the form $\lambda_{\mathbb{Q}}$ on $H \otimes \mathbb{Q}$ induced by λ and we obtain a rational symmetric bilinear form on $H^* \otimes \mathbb{Q}$. Restricting to $H^* \subset H^* \otimes \mathbb{Q}$, we obtain the form

$$\lambda^{-1} := (\hat{\lambda}_{\mathbb{Q}}^{-1})^*(\lambda_{\mathbb{Q}})|_{H^* \times H^*}: H^* \times H^* \rightarrow \mathbb{Q}.$$

Given a characteristic form (H, λ, α) and $x \in T$, let $\bar{x} \in H^*$ be such that $j(\bar{x}) = x$. We define the torsion form

$$b_{\lambda}: T \times T \rightarrow \mathbb{Q}/\mathbb{Z}, \quad (x, y) \mapsto \lambda^{-1}(\bar{x}, \bar{y}) \bmod \mathbb{Z},$$

and the quadratic refinement of b

$$q_{\lambda, \alpha}: T \rightarrow \mathbb{Q}/\mathbb{Z}, \quad x \mapsto \frac{\lambda^{-1}(\bar{x}, \bar{x}) + \lambda^{-1}(\bar{x}, \alpha)}{2} \bmod \mathbb{Z}. \quad (10)$$

The homogeneity defect of $q_{\lambda, \alpha}$ is exactly $j(\alpha)$.

We have following fundamental result of Wall:

Theorem 2.7 ([35, Theorem 6]). *For all torsion forms b and for every $q \in \mathcal{Q}^0(b)$, there is an even nondegenerate form (H, λ) and an isomorphism $q \cong \partial(H, \lambda, 0)$.*

We now state Milgram's theorem on the Gauss sums of homogeneous quadratic torsion forms.

Theorem 2.8 (Milgram [30, Theorem, p. 127, Appendix 4]). *Let $q \in \mathcal{Q}^0(b)$.*

- (i) $8A(q) \in \mathbb{Z}$.
- (ii) *If (H, λ) is an even nondegenerate integral form then $8A(q_{\lambda, 0}) \equiv \sigma(\lambda) \bmod 8$.*

Following Milgram's theorem, we can restate Nikulin's classification of homogeneous quadratic refinements of b as follows.

Theorem 2.9 ([32, Theorem 1.11.3]). *If $q_0, q_1 \in \mathcal{Q}^0(b)$ then q_0 is isomorphic to q_1 if and only if $A(q_0) = A(q_1)$.*

For general quadratic refinements of b we have the following results.

Proposition 2.10 ([5, Proposition 5.19]). *Let $q \in \mathcal{Q}(b)$ and suppose that $(G, q, b) = \partial(H, \lambda, \alpha)$. Then*

$$-A(q) = \frac{\lambda^{-1}(\alpha, \alpha) - \sigma(\lambda)}{8} \in \mathbb{Q}/\mathbb{Z}.$$

Theorem 2.11 ([5, Theorem 5.22], [10, Theorem 4.1]). *Two torsion forms $q_0, q_1 \in \mathcal{Q}(b)$ are isomorphic if and only if the following hold:*

- (i) *there is an automorphism $f: T \cong T$ of b such that $f(\beta_0) = \beta_1$,*
- (ii) $A(q_0) = A(q_1)$.

Remark 2.12. The proof of Theorem 2.9 in [32] and the proof of Theorem 2.11 in [5] both apply classification results for torsion forms and case by case checking. In contrast, the proof of Theorem 2.11 in [10] is short and general, with one elegant argument covering all cases.

2.4. Families of quadratic refinements. Let G be a finitely generated abelian group, p an element of $2G$, and b a torsion form on the torsion subgroup T , *i.e.* (G, b, p) belongs to the category \mathcal{Q}_b . Define

$$S_2 := \{h \in G : p_M - 2h \in T\},$$

and for $h \in S_2$ write $\beta_h := p_M - 2h$. Note that T acts simply transitively on S_2 by addition.

Definition 2.13. A *family of refinements* of (G, b, p) is a function $q^\circ : S_2 \rightarrow \mathcal{Q}(b)$, $h \mapsto q^h$, such that

- (i) the homogeneity defect of q^h is β_h , and
- (ii) $q^{h+t} = q^h_{-t}$ for any $t \in T$.

An isomorphism $F : G \rightarrow G'$ obviously maps $S_2 \rightarrow S'_2$, and F pulls back a family of quadratic refinements q'° on G' to one on G by setting

$$(F^\# q')^h := q'^{F(h)} \circ F|_T. \quad (11)$$

In this case q° and q'° are isomorphic via F . The orthogonal sum of two families of refinements (G_0, b_0, p_0) and (G_1, b_1, p_1) is the obvious family of refinements of $(G_0 \oplus G_1, b_0 \oplus b_1, p_0 \oplus p_1)$ as defined in Section 1.2 of the introduction. The negative of a family of refinements q° is the family of refinements $-q^\circ$ of $(Q, -b, p)$ defined by $(-q)^h = -q^h$. For later use we introduce the category \mathcal{Q}_q consisting of refinements with morphisms isomorphisms

$$\text{Ob}(\mathcal{Q}_q) = \{(G, q^\circ, p)\}.$$

Families of refinements as in Definition 2.13 are defined naturally on the boundaries of characteristic forms (H, λ, α) when λ is allowed to be degenerate. First define (G, b, p) as follows. Let $G := \text{coker}(\hat{\lambda})$ and $p := j(\alpha)$; that p is even should perhaps be seen as a consequence of existence of a characteristic element for any \mathbb{Z}_2 -valued bilinear form, degenerate or not. Let K be the radical $\ker(\lambda) \subset H$, and $R \subseteq H^*$ the annihilator of K . The form λ descends to a nondegenerate form on H/K , and $R \cong (H/K)^*$. Now $R/\text{Im}(\hat{\lambda}) \cong T$, so we obtain a torsion form b on T as in Section 2.3.

Next we define the induced family of quadratic refinements. For any $h \in S_2$, pick $m \in H^*$ such that $jm = h$ and set $\alpha_m = \alpha - 2m$. Then $j(\alpha_m) = \beta_h$, so $\alpha_m \in R$. Now α_m is characteristic for $(H/K, \lambda)$, and we let $q^h = q_{\lambda, \alpha_m}$ be the quadratic refinement of b defined in (10) in the previous subsection, *i.e.* if $y \in R$ and $ky = x$ then

$$q^h(y) := \frac{\lambda^{-1}(y, y) + \lambda^{-1}(\alpha_m, y)}{2} = \frac{\langle \hat{\lambda}^{-1}(y), y + \alpha_m \rangle}{2} \in \mathbb{Q}/\mathbb{Z}. \quad (12)$$

This is independent of the choice of m , since if $m' = m + \hat{\lambda}r$ then

$$\lambda^{-1}(2m', y) - \lambda^{-1}(2m, y) = 2\langle r, y \rangle \in 2\mathbb{Z}.$$

That (i) of Definition 2.13 is satisfied is immediate from $j\alpha_m = \beta_h$. Meanwhile, if $h' = h + t$ for some $t \in T$ then $j(m' - m) = t$, so

$$q^{h'}(x) - q^h(x) = \langle \hat{\lambda}^{-1}(y), m - m' \rangle = -b(x, t),$$

which shows that (ii) of Definition 2.13 holds. We denote this family of refinements q° by $\partial(H, \lambda, \alpha)$ and often refer to it as the *boundary* of (H, λ, α) . It is clear that an isomorphism of characteristic forms $E : (H_0, \lambda_0, \alpha_0) \cong (H_1, \lambda_1, \alpha_1)$ induces an isomorphism $\partial E : \partial(H_0, \lambda_0, \alpha_0) \cong \partial(H_1, \lambda_1, \alpha_1)$. It is also clear that the boundary of an orthogonal sum of characteristic forms is the orthogonal sum of the boundaries and that $\partial(H, -\lambda, \alpha) = -\partial(H, \lambda, \alpha)$.

A characteristic form (H, λ, α) is nonsingular if λ is, *i.e.* if the adjoint $\hat{\lambda} : H \rightarrow H^*$ is an isomorphism. Suppose that (H, λ, α) is a nonsingular characteristic form and that H_0 is some primitive subgroup of H , H_1 the λ -orthogonal subspace to H_0 , and λ_i, α_i the restrictions of λ and α to H_i . In this case we say that $(H_0, -\lambda_0, \alpha_0)$ and $(H_1, \lambda_1, \alpha_1)$ are *orthogonal in* (H, λ, α) . For the groups $G_i = \text{coker}(\hat{\lambda}_i)$ of the boundaries of $(H_i, \lambda_i, \alpha_i)$, the restriction maps $H^* \rightarrow H_i^*$ and the isomorphism $\hat{\lambda} : H \cong H^*$ give rise to homomorphisms $H \rightarrow H^* \rightarrow H_i^* \rightarrow G_i$ which induce isomorphisms

$$\Pi_i : H/(H_0 \oplus H_1) \cong G_i.$$

Hence we have the canonical isomorphism

$$F_\lambda := \Pi_1 \circ \Pi_0^{-1} : G_0 \cong G_1. \quad (13)$$

The following lemma is a routine calculation using (12) and the fact that (H, λ, α) is nonsingular: see [5, Lemma 3.10] for the case where $(H_i, \lambda_i, \alpha_i)$ are nondegenerate.

Lemma 2.14. *Let $(H_0, -\lambda_0, \alpha_0)$ and $(H_1, \lambda_1, \alpha_1)$ be orthogonal characteristic forms in the nonsingular characteristic form (H, λ, α) . The canonical isomorphism F_λ of (13) induces an isomorphism of the boundaries of $(H_0, -\lambda_0, \alpha_0)$ and $(H_1, \lambda_1, \alpha_1)$:*

$$F_\lambda^\# q_1^\circ = -q_0^\circ. \quad \square$$

As pointed out before, if W is a 3-connected coboundary of a closed spin 7-manifold M then $(FH^4(W, \partial W), \lambda_W, p_W)$ is a characteristic form. The associated boundary in \mathcal{Q}_b is precisely $(H^4(M), -b_M, p_M)$.

Definition 2.15. The *quadratic linking family* q_M° of M is the family of quadratic refinements of $(H^4(M), b_M, p_M)$ defined by the characteristic form $(FH^4(W, \partial W), -\lambda_W, p_W)$ of $-W$ for any 3-connected coboundary W or M .

That q_M° is independent of the choice of W and natural under homeomorphisms (in the sense that $(f^*)^\# q_{M_0}^\circ = q_{M_1}^\circ$ for any homeomorphism $f : M_0 \rightarrow M_1$) follows from Remark 2.1 and 2.3 and Lemmas 2.14: see Remark 2.4.

Remark 2.16. The sign in Definition 2.15 is introduced so that the linking form of q_M° is the usual linking form b_M . That this is the correct sign is shown in the proof of [1, Theorem 2.1]. The sign differs from [5, Definition 2.50] where the wrong sign was used.

Remark 2.17. If M is a rational homology sphere, then $0 \in S_2$ is a preferred element and we have the refinement $q_M := q_M^0$. It follows from [9, Definition 1.4 and Theorem 2.4] that q_M can be defined analytically using the eta invariant of the Dirac operator on M , twisted by appropriate quaternionic line bundles. As of the time of writing, we do not know of an intrinsic definition of the linking family q_M° when M is not a rational homotopy sphere.

Remark 2.18. The proof in Definition 2.15 that q_M° is a homeomorphism invariant relies on Remark 2.1 and Lemma 2.2. It is simpler than the proof given in [5, Theorem 6.1] which used the full apparatus of smoothing theory.

Notice, however, that smoothing theory and Theorem 1.2 imply that every 2-connected M with $H^2(M; \mathbb{Z}/2) \neq 0$ admits exotic self-homeomorphisms, by which we meant homeomorphisms which are not isotopic to piecewise linear homeomorphisms. Self-homotopy equivalences which are homotopic to exotic self-homeomorphisms were defined for rational homotopy spheres in [9, §2.b], see [9, Lemma 2.17].

Remark 2.19. Given a section $\sigma : G/T \rightarrow G$ of the projection $\pi : G \rightarrow G/T$, the image of σ is isomorphic to the free part of G , and there is a unique $k(\sigma) \in S_{d_\pi} \cap \text{Im}(\sigma)$. We can therefore define the family of quadratic refinements as a function on the set of sections $\text{Sec}(\pi)$ of π so that $q^\bullet : \text{Sec}(\pi) \rightarrow \mathcal{Q}(b)$, $q^\sigma := q^{\frac{d_\pi}{2} k(\sigma)}$. This presentation is relevant for considering connected-sum splittings of M and is discussed further in Section 3.2.

We could go in the direction of functions on elements on G and consider the restriction of a refinement q° to the subset $\tilde{S}_2 := \{h \in G : p_M = 2h\}$; for $h \in \tilde{S}_2$, q^h is a homogeneous refinement of b . We can define the quadratic linking families without mentioning quadratic linking functions, provided that we can always find a spin coboundary W with even intersection form (*i.e.* p_W even). The values of $h \mapsto q^h$ on all of S_2 can be recovered from the values on \tilde{S}_2 because of the transformation law, so the restriction to \tilde{S}_2 contains all the information of the previous notion of a linking family.

2.5. Gauss refinements. We can associate a further boundary invariant to a characteristic form which we refer to as a Gauss refinement of the family of quadratic refinements. Let $(G, p, b) \in \mathcal{Q}_b$,

i.e. a finitely generated abelian group G , an element $p \in 2G$ and $b : T \times T \rightarrow \mathbb{Q}/\mathbb{Z}$ a torsion form. Let d_π denote the greatest integer dividing p (or $d_\pi := 0$ if p is a torsion element), and let

$$S_{d_\pi} := \{k \in G : p_M - d_\pi k \in T\}. \quad (14)$$

Given $k \in S_{d_\pi}$ write $\beta_k := p_M - d_\pi k$ and note that T acts simply transitively on S_{d_π} by addition. We call a function $g : S_{d_\pi} \rightarrow \mathbb{Q}/\frac{d_\pi}{4}\mathbb{Z}$ (b, p) -linked if for all $k \in S_{d_\pi}$ and $t \in T$

$$g(k+t) = g(k) + \frac{\Delta(k, t)}{8}, \quad (15)$$

where

$$\Delta(k, t) := d_\pi^2 b(t, t) - 2d_\pi b(\beta_k, t) \in \mathbb{Q}/2d_\pi\mathbb{Z}. \quad (16)$$

Further, if $(G, q^\circ, p) \in \mathcal{Q}_q$, *i.e.* if q is a family of quadratic refinements of (G, b, p) , then we call a function $g : S_{d_\pi} \rightarrow \mathbb{Q}/\frac{d_\pi}{4}\mathbb{Z}$ a *Gauss refinement* of q if its mod $\frac{d_\pi}{4}\mathbb{Z}$ reduction is (b, p) -linked and in addition

$$g(k) = A(q^{\frac{d_\pi}{2}k}) \mod \mathbb{Z}. \quad (17)$$

(Note that if $k \in S_{d_\pi}$ then $\frac{d_\pi}{2}k \in S_2$, and $q^{\frac{d_\pi}{2}k}$ is a well-defined quadratic refinement of b .)

Now suppose that (H, λ, α) is a characteristic form. Given $k \in S_{d_\pi}$, pick $n \in H^*$ such that $jn = k$, and set $\alpha_n = p_W - d_\pi n$. Note that $j\alpha_n = \beta_k$, and that $\alpha_n \in R \cong (H/K)^*$ is a characteristic element for the intersection form on H/K . Let

$$g_H(k) := \frac{\lambda^{-1}(\alpha_n, \alpha_n) - \sigma(W)}{8} \in \mathbb{Q}/\frac{d_\pi}{4}\mathbb{Z}. \quad (18)$$

Lemma 2.20. g_H is well-defined, independent of the choices of n .

Proof. Replacing n by $n' := n + \hat{\lambda}r$, so $\alpha_{n'} = \alpha_n - d_\pi \hat{\lambda}r$, changes the value of $g_H(k)$ by

$$\frac{-2d_\pi \lambda^{-1}(\alpha, \hat{\lambda}r) + d_\pi^2 \lambda^{-1}(\hat{\lambda}r, \hat{\lambda}r)}{8} = \frac{d_\pi}{4} \left(-\langle r, \alpha_n \rangle + \frac{d_\pi}{2} \lambda(r, r) \right).$$

The last factor is an integer, and it is even when d_π is not divisible by 4 (*i.e.* when $\tilde{d}_\pi = 2d_\pi$) because α_n is characteristic. \square

Lemma 2.21. $g_H(k) = -A(q^{\frac{d_\pi}{2}k}) \mod \mathbb{Z}$.

Proof. The α_n used in the definition of $g_H(k)$ co-incides with the α_m used in the definition of $g^{\frac{d_\pi}{2}k}$ in (12). Since α_n is characteristic for λ , the lemma is immediate from Proposition 2.10. \square

Finally we check the transformation law.

Lemma 2.22.

$$g_H(k+t) - g_H(k) = \frac{\Delta(k, t)}{8} \mod \frac{d_\pi}{4}.$$

Proof. $\alpha_{n'} - \alpha_n = -d_\pi(n' - n)$, and $j(n' - n) = t$. Hence

$$\lambda^{-1}(\alpha_{n'}, \alpha + n') - \lambda^{-1}(\alpha_n, \alpha_n) = -2d_\pi \lambda^{-1}(\alpha_n, n' - n) + d_\pi^2 \lambda^{-1}(n' - n, n' - n) = \Delta(k, t) \mod 2d_\pi. \quad \square$$

Lemma 2.21 and 2.22 say precisely that g_H is a Gauss refinement of q in the sense defined above.

Remark 2.23. We could make an analogy with factors of automorphy of automorphic forms and think of Δ as a “term of automorphy”. For any linked functions to exist is clearly equivalent to the cocycle condition

$$\Delta(k, s+t) = \Delta(k+s, t) + \Delta(k, s), \quad (19)$$

which can be checked directly from the definition in (15). The difference of two functions with the same term of automorphy is invariant under the T action; since T acts transitively on S_{d_π} that simply means that the difference between two (p, b) -linked functions is a constant in $\mathbb{Q}/2d_\pi\mathbb{Z}$.

We define $\tilde{\Delta}(k, t) \in \mathbb{Q}/2\tilde{d}_\pi\mathbb{Z}$ by

$$\tilde{\Delta}(k, t) = \Delta(k, t) \pmod{2d_\pi}, \quad (20a)$$

$$\tilde{\Delta}(k, t) = 8q^{\frac{d_\pi}{2}k}(\frac{d_\pi}{2}t) \pmod{8}. \quad (20b)$$

If d_π is not divisible by 4 then in order for $\tilde{\Delta}$ to be well-defined we need to check that the mod 4 reductions agree, which follows from

$$2q^{\frac{d_\pi}{2}k}(x) = b(x, x) + b(\beta_k, x).$$

Similarly, for $4 \mid d_\pi$ we deduce that (20a) implies (20b) from

$$q^{\frac{d_\pi}{2}k}(2x) = 2b(x, x) + b(\beta_k, x).$$

Now any Gauss refinement g of q° satisfies the transformation rule

$$g(k+t) - g(k) = \frac{\tilde{\Delta}(k, t)}{8}, \quad (21)$$

since $A(q^{\frac{d_\pi}{2}(k+t)}) - A(q^{\frac{d_\pi}{2}k}) = q^{\frac{d_\pi}{2}k}(\frac{d_\pi}{2}t) \in \mathbb{Q}/\mathbb{Z}$ by (9).

The difference between two Gauss refinements of the same linking family is simply a constant in $\mathbb{Z}/\frac{\tilde{d}_\pi}{4}\mathbb{Z}$. The difference between Gauss refinements of different linking families need not be constant mod \mathbb{Z} if d_π is not divisible by 4 (since then $\tilde{\Delta}$ can depend on q), and in any case need not take integer values.

An isomorphism $F : G \rightarrow G'$ with $F(p) = p'$ maps $S_{d_\pi} \rightarrow S'_{d_\pi}$. If $F^\#b' = b$ then we have $\Delta'(F(k), F(t)) = \Delta(k, t)$ for all $k \in S_{d_\pi}$ and $t \in T$, so if $g : S'_{d_\pi} \rightarrow \mathbb{Q}/\frac{d_\pi}{4}\mathbb{Z}$ is a linked function then so is

$$F^\#g := g \circ F. \quad (22)$$

Similarly, if g is a Gauss refinement of a linking family q° on G' , then $F^\#g$ is a Gauss refinement of the linking family $F^\#q^\circ$ on G .

We next consider the Gauss refinements of orthogonal characteristic forms. We recall that if (H, λ, α) is nonsingular and $H_0 \subset H$ is primitive with orthogonal complement H_1 , then for the characteristic forms $(H_i, \lambda_i, \alpha_i)$ defined by restriction from (λ, α) , there is a canonical isomorphism $F_\lambda : \partial(H_0, -\lambda_0, \alpha_0) \cong \partial(H_1, \lambda_1, \alpha_1)$.

Lemma 2.24. *Let $(H_0, -\lambda_0, \alpha_0)$ and $(H_1, \lambda_1, \alpha_1)$ be orthogonal characteristic forms in the nonsingular characteristic form (H, λ, α) . The canonical isomorphism F_λ of (13) induces an isomorphism*

$$F_\lambda^\# g_{H_1} - g_{H_0} = \frac{\lambda(\alpha, \alpha) - \sigma(\lambda)}{8} \pmod{\frac{\tilde{d}_\pi}{4}}.$$

Proof. Note that since (H, λ) is nonsingular, $(\hat{\lambda})^{-1} : H^* \cong H$ is an isomorphism from λ^{-1} to λ . Also, we have homomorphisms $H^* \rightarrow H_i^* \rightarrow G_i$ where we recall that $G_i = \text{coker}(\hat{\lambda}_i)$. Pick a $k \in S_{d_\pi}(G_0)$, and then pick $n \in H^*$ whose image in G_0 equals k (the set-up means that the image of n in G_1 is $F_\lambda(k)$). Let n_i be the image of n in H_i^* , and set $\alpha_{n_i} := \alpha_i - d_\pi n_i \in R_i$ as in the definition of g_{H_i} . We need to show that

$$\lambda^{-1}(\alpha, \alpha) = \lambda_1^{-1}(\alpha_{n_1}, \alpha_{n_1}) - \lambda_0^{-1}(\alpha_{n_0}, \alpha_{n_0}) \pmod{2\tilde{d}_\pi}. \quad (23)$$

The image of $\alpha - d_\pi n$ in $H^* \otimes \mathbb{Q}$ can be written as a sum $\hat{\lambda}_0(\gamma_0) + \hat{\lambda}_1(\gamma_1)$ where $\gamma_i \in H_i \otimes \mathbb{Q}$ and $\hat{\lambda}_i(\gamma_i) = \alpha_{n_i}$. Thus

$$\begin{aligned} \lambda^{-1}(\alpha, \alpha) &= \lambda_1(\gamma_1, \gamma_1) - \lambda_0(\gamma_0, \gamma_0) + 2d_\pi \lambda^{-1}(n, \alpha - d_\pi n) + d_\pi^2 \lambda^{-1}(n, n) \\ &= \lambda_1^{-1}(\alpha_{n_1}, \alpha_{n_1}) - \lambda_0^{-1}(\alpha_{n_0}, \alpha_{n_0}) \pmod{2\tilde{d}_\pi}; \end{aligned}$$

that equality holds mod $4d_\pi$ when d_π is not divisible by 4 follows from $\alpha - d_\pi n$ being a characteristic element for λ . \square

Remark 2.25. We call a characteristic form (H, λ, α) *neutral* if it is nonsingular and $\lambda(\alpha, \alpha) = \sigma(\lambda)$. Say that two characteristic forms are *neutrally isomorphic* if they become isomorphic after addition of neutral forms (so this is a sharper condition than stable isomorphism). Lemma 2.24 implies that Gauss refinements are invariant under neutral isomorphism. The gluing and splitting arguments for characteristic forms reviewed in Section 3.2, in particular Theorem 3.3, can be used to show that characteristic forms are classified to neutral isomorphism. by their boundary distillations (G, q°, g, p) classify

2.6. The generalised Eells–Kuiper invariant. Let M be a spin 7-manifold and W a 3-connected coboundary of M . Let g_W be the Gauss refinement of $(H^4(M), q_M^\circ, p_M)$ defined by the characteristic form $(H^4(W, \partial W), \lambda_W, p_W)$: notice that we *did not use* $-W$ as we did in defining the linking family of M . If $f : M_0 \rightarrow M_1$ is a spin homeomorphism the $X := (-W_0) \cup_f W_1$ a closed is a closed topological spin 8-manifold and Lemmas 2.3 and 2.24 imply

$$g_{W_1} - (f^*)^\# g_{W_0} = \frac{p_X^2 - \sigma(X)}{8} \mod \frac{\hat{d}_\pi}{4} \mathbb{Z}. \quad (24)$$

If f is a diffeomorphism then X is smooth, the RHS of (24) equals $28\hat{A}(X)$, and $\hat{A}(X)$ is an integer. Letting

$$\hat{d}_\pi := \gcd\left(\frac{\tilde{d}_\pi}{4}, 28\right) \quad (25)$$

as in the introduction, it follows that

$$\begin{aligned} \mu_M : S_{d_\pi} &\rightarrow \mathbb{Q}/\hat{d}_\pi \mathbb{Z}, \\ \mu_M &:= g_W \mod \hat{d}_\pi \end{aligned} \quad (26)$$

is independent of the choice of W , and natural under diffeomorphisms: If $f : M_0 \rightarrow M_1$ is a spin diffeomorphism then $(f^*)^\# \mu_{M_0} = \mu_{M_1}$.

From this definition of μ_M via the characteristic form of a 3-connected coboundary we recover the description in the introduction. This description is valid also when W is not 3-connected, although g_W might not be defined on all of S_{d_π} in this case. This point can be seen as a special case of Proposition 2.33 below.

Remark 2.26. Analogously to Remark 2.19, we can define Gauss refinements (and μ_M) as functions of sections $\sigma : G/T \rightarrow G$ rather than on S_{d_π} , $g_W(\sigma) := g_W(k(\sigma))$. Then $g_W(\sigma) = A(q^\sigma) \mod \mathbb{Z}$, and the transformation rule (15) can also be rewritten in these terms.

Remark 2.27. Recall Remark (1.1) saying that if p_M is torsion then $S_{d_\pi} = T$ contains the distinguished element 0 and $\frac{1}{28}\mu_M(0) \in \mathbb{Q}/\mathbb{Z}$ recovers the original Eells–Kuiper invariant $\mu(M)$.

Although defined extrinsically using spin co-boundaries, the original Eells–Kuiper invariant $\mu(M)$ was shown by Donnelly [11, Theorem 4.2] to have an intrinsic definition in terms of the eta invariant of the Dirac operator of M . As of the time of writing, we do not know of an intrinsic definition of the generalised Eells–Kuiper invariant when $p_M \neq 0 \in H^4(M; \mathbb{Q})$.

For further information about the role of eta invariants in the classification of 7-manifolds, we refer the reader to [14, §4].

Remark 2.28. In [5, §4.4], a pair of characteristic forms (H, λ, α) are called *smoothly equivalent* if they become isomorphic after addition of non-singular forms with $\lambda(\alpha, \alpha) \equiv \sigma(\lambda) \mod 224$ (so this is a weakening of the notion of neutral equivalence from Remark 2.25). In algebraic terms, the definition of the generalised Eells–Kuiper invariant can be used to show that the mod 28 Gauss refinement of M , $(H^4(M), q_M^\circ, \mu_M, p_M)$, is a complete invariant of the smooth equivalence class of the characteristic form $(H^4(W, \partial W), \lambda_W, p_W)$ of a 3-connected coboundary for M . Hence Theorem 1.3 is a development [5, Theorem 4.9] in dimension 7 where 2-connected 7-manifolds are classified up to diffeomorphism by the smooth equivalence class of the characteristic form of a 3-connected coboundary.

2.7. The computation of μ_M via spin^c coboundaries. Inspired by the s_1 invariant of Kreck-Stolz [24], we derive an expression for μ_M in terms of coboundaries that are not necessarily spin (never mind 3-connected) but just spin^c .

A principal spin^c bundle is equivalent to a real vector bundle E together with a complex line bundle L such that $c_1(L) = w_2(E) \bmod 2$. We can associate to this the characteristic classes

$$\begin{aligned} z &:= c_1(L), \\ \check{p} &:= p(E \oplus L), \\ \hat{p} &:= \check{p} + z^2. \end{aligned}$$

So $2\check{p} = p_1(E \oplus L) = p_1(E) - z^2$, and $2\hat{p} = p_1(E) + z^2$. Recall that any U -bundle has a natural spin^c structure: if E is a complex vector bundle we may take $L := \det E$.

Lemma 2.29.

- (i) \check{p} and z^2 form a basis for $H^4(B\text{Spin}^c)$.
- (ii) $\check{p}(E) = -c_2(E)$ for any complex bundle E .
- (iii) $\check{p}(E) = w_4(E) \bmod 2$ for any spin^c bundle E .

Proof. Observe that $\text{Spin}^c/U \cong \text{Spin}/SU$ 5-connected implies $\pi^* : H^4(B\text{Spin}^c) \rightarrow H^4(BU)$ and $\pi^* : H^2(B\text{Spin}^c) \rightarrow H^2(BU)$ are isomorphisms (with both \mathbb{Z} and \mathbb{Z}_2 coefficients). The image of z is patently c_1 .

We know that $H^4(BU)$ has basis c_2, c_1^2 . Because there is no 2-torsion, $2\pi^*\check{p} = p_1 - (\pi^*z)^2 = (-2c_2 + c_1^2) - c_1^2$ implies $\pi^*\check{p} = -c_2$, proving (i) and (ii).

The isomorphism on $H^4(-; \mathbb{Z}_2)$ implies that it suffices to check that (iii) holds when E is complex. But that follows from (ii). \square

Corollary 2.30. *If X is a compact spin^c 8-manifold then \hat{p}_X is characteristic for the intersection form λ_X of X .*

Proof. Lemma 2.29 gives

$$\hat{p} = w_4 + w_2^2 \bmod 2.$$

If X is closed, then Wu's formula implies that for any closed orientable manifold X the fourth Wu class is $v_4(X) = w_4(X) + w_2(X)^2$. The compact case follows from the closed case, as in the proof of Lemma 2.2(iii). \square

Lemma 2.31. *If X is a closed spin^c 8-manifold then the Dirac operator of the fundamental complex spinor bundle has*

$$28 \text{ind } \not{D}^+ = \frac{\hat{p}_X^2 - \sigma(X)}{8} - \frac{5z^2\hat{p}_X}{12} + \frac{z^4}{4}. \quad (27)$$

Proof. [25, Theorem D.15] expresses $\text{ind } \not{D}^+$ as the integral of $\exp(\frac{z}{2}) \hat{A}(X)$, whose degree 8 part expands to

$$\frac{-4p_2 + 7p_1^2}{2^7 \cdot 45} - \frac{z^2 p_1}{24 \cdot 8} + \frac{z^4}{24 \cdot 16} = \frac{p_1^2}{2^7 \cdot 7} - \frac{L}{2^5 \cdot 7} - \frac{z^2 p_1}{2^6 \cdot 3} + \frac{z^4}{2^7 \cdot 3}.$$

Then substitute $p_1 = 2\hat{p} - z^2$ to obtain (27). \square

Now suppose M is a spin 7-manifold and W a spin^c coboundary, such that the restriction of $z \in H^2(W)$ to M is trivial. Then z has a pre-image $\bar{z} \in H^2(W, M)$, and $\bar{z}^2 \in H^4(W, M)$ is independent of the choice of \bar{z} .

Definition 2.32. Given $k \in S_{d_\pi}$, suppose there is $n \in H^*$ such that $jn = k$. Let $\hat{\alpha}_n := \hat{p}_W - d_\pi n$, and

$$g_W^c(k) := \frac{\lambda^{-1}(\hat{\alpha}_n, \hat{\alpha}_n) - \sigma(W)}{8} - \frac{5\bar{z}^2 \hat{p}_W}{12} + \frac{z^4}{4} \in \mathbb{Q}/\frac{i}{4}\mathbb{Z}.$$

If $z = 0$ then of course $g_W^c = g_W$. The proof that $g_W^c(k)$ does not depend on the choice of n is analogous to Lemma 2.20, using that $\hat{\alpha}$ is characteristic for intersection form λ_W .

Proposition 2.33. *Let (W_1, z_1) be a spin^c coboundary of M and $j_1: H^4(W_1) \rightarrow H^4(M)$ the natural homomorphism. For all $k, k' \in S_{d_\pi} \cap j_1(H^4(W_1))$ we have:*

- (i) $g_{W_1}^c(k) = \mu_M(k) \bmod 28$ for all $k \in S_{d_\pi} \cap j_1(H^4(W_1))$;
- (ii) *The defined values of $g_{W_1}^c$ satisfy the transformation rule (21), i.e. if $k, k' \in S_{d_\pi} \cap j_1(H^4(W_1))$ then $g_{W_1}^c(k') = g_{W_1}^c(k) + \frac{\tilde{\Delta}(k'-k, t)}{8}$.*

Proof. For part (i), take W_0 to be a 3-connected coboundary for M , and let $X := (-W_0) \cup_{\text{Id}_M} W_1$. Then X is a smooth spin^c manifold, possibly with more than one choice of $z \in H^2(X)$ restricting to z_1 on W_1 and 0 on W_0 . While we do not trouble ourselves with separating the algebra from the topology in this case, we essentially adapt the proof of Lemma 2.24 to show

$$g_{W_1}^c - g_{W_0} = \frac{\hat{p}_X^2 - \sigma(X)}{8} - \frac{5z^2 \hat{p}_X}{12} + \frac{z^4}{4} \bmod \frac{\tilde{d}_\pi}{4} \mathbb{Z}. \quad (28)$$

Since the RHS equals 28 ind \tilde{D}^+ by Lemma 2.31, while $g_W = \mu_M \bmod \gcd(28, \frac{\tilde{d}_\pi}{4})$ by definition, the result then follows.

Pick some $n_1 \in H^4(W_1)$ such that $j_1 n_1 \in S_{d_\pi}$ as in Definition 2.32. Because W_0 is 3-connected there is some $n \in H^4(X)$ whose restriction to W_1 equals n_1 . Then $\hat{p}_X - d_\pi n$ is a sum of push-forwards of $\gamma_i \in H^4(W_i, M; \mathbb{Q})$, and $\hat{\lambda}_{W_i} \gamma_i = \hat{\alpha}_i$. Meanwhile, note that regardless of the choice of z , $z^2 \in H^4(X)$ is the push-forward of $\tilde{z}_1^2 \in H^4(W_1, M)$. Hence

$$\begin{aligned} \hat{p}_X^2 - \frac{10z^2 \hat{p}_X}{3} + 2z^4 &= \gamma_1^2 + \gamma_0^2 + 2d_\pi n(\hat{p}_X - d_\pi n) + d_\pi^2 n^2 - \frac{10}{3} \tilde{z}_1^2 \hat{p}_{W_1} + 2z_1^4 \\ &= \lambda_{W_1}^{-1}(\hat{\alpha}_1, \hat{\alpha}_1) - \frac{10}{3} \tilde{z}_1^2 \hat{p}_{W_1} + 2z_1^4 - \lambda_{W_0}^{-1}(\hat{\alpha}_0, \hat{\alpha}_0) \bmod 2\tilde{d}_\pi. \end{aligned}$$

The fact that the equality holds $\bmod 4d_\pi$ when d_π is not divisible by 4 is due to $\hat{p}_X - d_\pi n$ being a characteristic element for the intersection form on X .

Part (ii) follows from (28) since g_{W_0} satisfies (21) and the RHS of (28) is constant. \square

The consequence of (ii) is that we can extend g_W^c to a well-defined Gauss refinement as long as $S_{d_\pi} \cap j(H^4(W))$ is non-empty.

3. THE CLASSIFICATION OF 2-CONNECTED 7-MANIFOLDS

In this section we classify closed smooth spin 2-connected 7-manifolds M up to spin diffeomorphism. Recall that a homotopy 7-sphere Σ is a spin manifold which is homotopy equivalent to S^7 . In Section 3.1 we recall that by definition an almost diffeomorphism $f: M \cong N$ defines a diffeomorphism $f: M \cong N \sharp \Sigma$, for some homotopy sphere Σ . In Section 3.2 we relate the algebra of Section 2 to the algebra used in [5] and so give the almost diffeomorphism classification of 2-connected M in terms of their linking families $(H^4(M), q_M^\circ, \mu_M, p_M)$.

With the almost diffeomorphism classification in hand, we are lead to consider the inertia group of M . This is the group of homotopy spheres Σ such that $M \sharp \Sigma \cong M$. In Section 3.3 we establish basic facts about the inertia group of M , the reactivity of M and the relationship between reactivity and inertia. We also construct an important family of almost diffeomorphisms $f: M \cong M$ in Proposition 3.10. The almost diffeomorphisms of Proposition 3.10 allow us to show that the generalised Eells-Kuiper invariant of M , μ_M , precisely measures the gap between the almost diffeomorphism classification and the diffeomorphism classification. In Section 3.4 we prove that the mod 28 Gauss refinement of M , $(H^4(M), q_M^\circ, \mu_M, p_M)$, is a complete invariant diffeomorphisms.

3.1. Almost diffeomorphisms. In this subsection we briefly review the almost diffeomorphism category in dimension 7. We shall return to consider almost diffeomorphisms in more detail in Section 3.3. An almost diffeomorphism $f: M_0 \cong M_1$ is a spin homeomorphism which is smooth except perhaps at a finite number of singular points $\{m_0, \dots, m_a\} \subset M_0$. Notice that *we do not require f to be non-smooth at m_i* , but we rather *allow* it. In this case M_0 and M_1 are called almost diffeomorphic. If $M = M_0 = M_1$, then we can compose almost diffeomorphisms and we define

$$\text{ADiff}(M) := \{f \mid f: M \cong M\},$$

to be the group of almost diffeomorphisms of M .

Let $f_i, i = 0, 1$, be almost diffeomorphisms with singular points $\{m_1, \dots, m_{a_0}\}$ and $\{n_1, \dots, n_{a_1}\}$. A pseudo-isotopy between almost diffeomorphisms is a spin homeomorphism

$$F: M_0 \times I \cong M_1 \times I$$

such that $F|_{M_0 \times \{i\}} = f_i$ which is smooth except perhaps along $X \subset M_0 \times I$, where X is a smoothly embedded 1-manifold with boundary $\partial X = \{n_1, \dots, n_{a_0}\} \sqcup \{m_1, \dots, m_{a_1}\}$ and precisely $(a_0 + a_1)/2$ connected components. In particular, X has no closed components in the interior of $M_0 \times I$ and a_0 and a_1 have the same parity. Pseudo-isotopy defines an equivalence relation on almost diffeomorphisms which is compatible with composition and hence if $M = M_0 = M_1$ we define

$$\tilde{\pi}_0 \text{ADiff}(M) := \{f \mid f: M \cong M\},$$

to be the group of pseudo-isotopy classes of almost diffeomorphisms of M .

Later we shall wish to take the connected sum of two almost diffeomorphisms if $f: M_0 \cong M_1$ and $g: N_0 \cong N_1$. By the disc theorem [33, Theorem 5.5], we can assume after isotopy that there is embeddings $\iota_M: D^7 \rightarrow M_0$ and $\iota_N: D^7 \rightarrow N_0$ with images disjoint from the singular points of f and g and with $f|_{\iota_M(D^7)} = g|_{\iota_N(D^7)}$. Taking the connected sum of M_0 and N_0 along ι_M and ι_N we can then extend f and g to form an almost diffeomorphism

$$f \# g: M_0 \# N_0 \cong M_1 \# N_1.$$

3.2. The almost diffeomorphism classification. In this subsection we show how Theorem 1.2 follows from the classification results of [5]. The almost diffeomorphism classification given in [5] used a different but closely related definition of a quadratic linking family. We begin by explaining the relationship between the two definitions of linking family and showing that Theorem 1.2 is equivalent to [5, Theorem B]. We then describe the main ideas of the proof of [5, Theorem B] and interpret linking families in terms of connected sum splittings. Throughout this subsection M is 2-connected, $G = H^4(M)$ with torsion subgroup $T \subset G$ and free quotient $F = G/T$.

Let us start with some elementary algebra for the group G . Let $\iota: T \rightarrow G$ be the inclusion of the torsion subgroup and let $\pi: G \rightarrow F$ be the canonical projection. We let $\text{Sec}(\pi) := \{\sigma: F \rightarrow G\}$ be the set of sections $\pi: G \rightarrow F$ and we let $\text{Proj}(\iota) := \{\tau: G \rightarrow T\}$ be the set of projections over $\iota: \tau \circ \iota = \text{Id}_T$. The sets $\text{Sec}(\pi)$ and $\text{Proj}(\iota)$ are in bijection by mapping $\sigma \mapsto \tau_\sigma$ where $\text{Im}(\sigma) = \ker(\tau_\sigma)$. Both sets admit simple transitive actions of $\text{hom}(F, T)$ via addition of functions: For $\phi \in \text{hom}(F, T)$ $f \in F$ and $g \in G$ we have

$$(\sigma + \phi)(f) = \sigma(f) + \phi(f) \quad \text{and} \quad (\tau + \phi)(g) = \tau(g) + \phi(\pi(g)).$$

Notice that $\tau_{\sigma+\phi} = \tau_\sigma - \phi$.

Remark 3.1. The action of $\text{hom}(F, T)$ on $\text{Sec}(\pi)$ used above differs by a sign from the corresponding action in [5, p. 39].

Let (G, b, p) be a base so that b is a torsion form on T and $p \in 2G$. Recall that $\mathcal{Q}(b)$ is the set of refinements of b and given $q \in \mathcal{Q}(b)$, let us write $\beta(q)$ for the homogeneity defect of q : see Section 2.3. In [5, Definition 2.39] a quadratic linking family on a base (G, b, p) was defined as a function

$$q^\bullet: \text{Sec}(\pi) \rightarrow \mathcal{Q}(b)$$

such that for all $\sigma \in \text{Sec}(\pi)$ and for all $\phi \in \text{hom}(G, T)$,

$$q^{\sigma+\phi} = q_{-\phi(\pi(p))}^\sigma \quad \text{and} \quad \beta(q^\sigma) = \tau_\sigma(p).$$

We explain the topological significance of these conditions below, focussing for now on the algebra.

In this paper we work with refinements which are functions on S_2 . To relate q^\bullet to the linking family $q^\circ: S_2 \rightarrow \mathcal{Q}(b)$ defined in 2.13 we start from the set $S_{d_\pi} \subseteq G$. Multiplication by $\frac{d_\pi}{2}$ gives a map $S_{d_\pi} \rightarrow S_2$ and we define $\hat{S}_2 = \frac{d_\pi}{2} S_{d_\pi} \subset S_2$. There is a function

$$\text{Sec}(\pi) \rightarrow S_{d_\pi}, \quad \sigma \mapsto k(\sigma) \in \text{Im}(\sigma) \cap S_{d_\pi}$$

and given a refinement $q^\circ: S_2 \rightarrow \mathcal{Q}(b)$ we define

$$q^\bullet: \text{Sec}(\pi) \rightarrow \mathcal{Q}(b), \quad q^\sigma := q^{\frac{d\pi}{2}k(\sigma)}. \quad (29)$$

Conversely, given $q^\bullet: \text{Sec}(\pi) \rightarrow \mathcal{Q}(b)$ we define

$$q^\circ: \widehat{S}_2 \rightarrow \mathcal{Q}(b), \quad q^{\frac{d\pi}{2}k(\sigma)} := q^\sigma, \quad (30)$$

and extend to all of S_2 by the transformation rule of Definition 2.13 (ii). Indeed the transformation rule ensures that a refinement q° is determined by its value, q^h , for a single value of $h \in S_2$ and hence we have

Lemma 3.2. *The mappings $q^\circ \mapsto q^\bullet$ and $q^\bullet \mapsto q^\circ$ of (29) and (30) define inverse equivalences of categories between linking families defined on S_2 and linking families defined on $\text{Sec}(\pi)$. \square*

Proof of Theorem 1.2. Let M be 2-connected, let q_M° be the linking family of M as defined in Definition 2.15 and let $-q_M^\bullet$ be the linking family of M as defined in [5, Definition 2.39] (we have introduced the sign to correct the mistake in [5, Definition 2.50]: see Remark 2.16.) Comparing these definitions, we see that for each $\sigma \in \text{Sec}(\pi)$

$$q_M^{\frac{d\pi}{2}k(\sigma)} = q_M^\sigma. \quad (31)$$

Now [5, Theorem B] states that all linking families defined on $\text{Sec}(\pi)$ arise as the quadratic linking families of 2-connected M and that any isomorphism of linking families defined on $\text{Sec}(\pi)$ is realised by an almost diffeomorphism. Hence Theorem 1.2 follows by combining [5, Theorem B], (31) and Lemma 3.2. \square

We now explain the proof of [5, Theorem B]. Recall that every 2-connected M is the boundary of a 3-connected W and that the characteristic form of W , $(H^4(W, \partial W), \lambda_W, \alpha_W)$ is a complete diffeomorphism invariant by [5, Corollary 2.5]. A foundational theorem of Wilkens [37, Theorem 3.2] (see [5, Theorem 2.24]), states that for any diffeomorphism $f: \partial W_0 \cong \partial W_1$, there are W_2 and W_3 with $\partial W_2 = \partial W_3 = S^7$ and a diffeomorphism $F: W_0 \natural W_2 \cong W_1 \natural W_3$ extending f : here \natural denotes boundary connected sum. Now the boundary of W is a homotopy sphere, if and only if λ_W is nonsingular which by definition means that $(H^4(W, \partial W), \lambda_W, \alpha_W)$ is nonsingular. Hence we say that two characteristic forms are *stably isomorphic* if they become isomorphic after addition of nonsingular characteristic forms. The above discussion shows that classifying 2-connected 7-manifolds up to almost diffeomorphism is equivalent to classifying characteristic forms up to stable isomorphism. This was achieved in [5, Theorem 3.4] by extending ideas of Wall [36, Theorem p.156] from the setting of even forms to the setting of characteristic forms. The point is that an isomorphism $F: \partial(H_0, \lambda_0, \alpha_0) \cong \partial(H_1, \lambda_1, \alpha_1)$ of the boundaries of characteristic forms can be used to glue them together to obtain a nonsingular characteristic form

$$(H_0, -\lambda_0, \alpha_0) \cup_F (H_1, \lambda_1, \alpha_1).$$

It is then possible to explicitly write down an isomorphism of characteristic forms

$$E: (H_0, \lambda_0, \alpha_0) \oplus ((H_0, -\lambda_0, \alpha_0) \cup_F (H_1, \lambda_1, \alpha_1)) \cong (H_1, \lambda_1, \alpha_1) \oplus ((H_0, -\lambda_0, \alpha_0) \cup_{\text{Id}} (H_0, \lambda_0, \alpha_0)),$$

such that $\partial E = F$. Combined with Lemma 2.14, these methods give the following theorem, which is a refinement of a special case of [5, Theorem 3.4].

Theorem 3.3 (cf. [5, Theorem 3.4]). *Let $(H_i, \lambda_i, \alpha_i)$, $i = 0, 1$, be two characteristic forms. The following are equivalent:*

- (i) *There is an isomorphism of linking families $F: \partial(H_0, \lambda_0, \alpha_0) \cong \partial(H_1, \lambda_1, \alpha_1)$;*
- (ii) *There are nonsingular characteristic forms $(H_j, \lambda_j, \alpha_j)$, $j = 2, 3$, and an isomorphism*

$$E: (H_0, \lambda_0, \alpha_0) \oplus (H_2, \lambda_2, \alpha_2) \cong (H_1, \lambda_1, \alpha_1) \oplus (H_3, \lambda_3, \alpha_3)$$

such that $\partial E = F$;

- (iii) *There is a nonsingular characteristic form (H, λ, α) containing $(H_0, -\lambda_0, \alpha_0)$ and $(H_1, \lambda_1, \alpha_1)$ are orthogonal summands with canonical isomorphism $F_\lambda = F: \partial(H_0, \lambda_0, \alpha_0) \cong \partial(H_1, \lambda_1, \alpha_1)$.*

Remark 3.4. By Lemma 3.2 the statement of Theorem 3.3 and the discussion before it applies equally well to linking families defined over $\text{Sec}(\pi)$ and linking families defined over S_2 .

Proof of Theorem 3.3. This follows from [5, Lemma 3.12] and the proof of [5, Theorem 3.4]. \square

We now explain the topological significance of q_M° . By Theorem 1.2, 2-connected rational homotopy spheres M with torsion linking for (T, b) are classified up to almost diffeomorphism by the refinement $q_M \in \mathcal{Q}(b)$. We shall write $M(q)$ for any rational homotopy sphere with refinement q . In contrast to rational homotopy spheres where $G = T$, the canonical examples of 2-connected M with $H^4(M)$ torsion free are the manifolds

$$M(F, d_\pi) := (S^3 \tilde{\times}_{d_\pi} S^4) \#_{b-1} (S^3 \times S^4), \quad (32)$$

where $F \cong \mathbb{Z}^b$, $S^3 \tilde{\times}_{d_\pi} S^4$ is the S^3 -bundle over S^4 with trivial Euler class and first Pontrjagin class d_π times a generator of $H^4(S^4)$. In the general case we shall be interested in connected sum almost splittings of M which are almost diffeomorphisms

$$f: M \cong M(q^f) \# M(F, d_\pi),$$

where q^f is a certain refinement of b_M . We call two connected sum almost splittings f_0 and f_1 H^* -equivalent if there is an almost diffeomorphism $g: M \cong M$ with $H^*(g) = \text{Id}$ and if there are almost diffeomorphisms $g_T: M(q_0) \cong M(q_1)$ and $g_F: M(F, d_\pi) \cong M(F, d_\pi)$ such that the following diagram commutes up to pseudo-isotopy.

$$\begin{array}{ccc} M & \xrightarrow{f_0} & M(q^{f_0}) \# M(F, d_\pi) \\ \downarrow g & & \downarrow g_T \# g_F \\ M & \xrightarrow{f_1} & M(q^{f_1}) \# M(F, d_\pi). \end{array}$$

We define $\text{ASplit}(M) := \{[f]\}$ to be the set of H^* -equivalence classes of almost splittings of M and note that there is a well-defined map

$$\text{ASplit}(M) \mapsto \text{Sec}(\pi), \quad [f] \mapsto \sigma(f),$$

where $\text{Im}(\sigma(f)) = f^* H^4(M(F, d_\pi))$. The following theorem is implicit in [5, Definition 2.50].

Theorem 3.5. *Let M have linking family $q_M^\bullet: \text{Sec}(\pi) \rightarrow \mathcal{Q}(b)$. For each $\sigma \in \text{Sec}(\pi)$ there is a unique H^* -equivalence class of almost splitting*

$$f_\sigma: M \cong M(q_M^\sigma) \# M(F, d_\pi).$$

Consequently the map $\text{ASplit}(M) \rightarrow \text{Sec}(\pi)$ is a bijection.

Proof. Let W be a 3-connected coboundary of M with characteristic form $(H^4(W, \partial W), \lambda_W, \alpha_W) = (H, \lambda, \alpha)$. We recall from the definition of the linking family defined by W in (12), that there are orthogonal splittings of (H, λ, α)

$$\psi: (H, \lambda, \alpha) \cong (R, \lambda_R, \alpha_\psi) \oplus (F, 0, \alpha|_F),$$

where $(R, \lambda_R, \alpha_\psi)$ is nondegenerate. For every such splitting ψ , the classification of 3-connected coboundaries (see [5, Corollary 2.5]) implies that there is a corresponding boundary connected sum splitting $g_\psi: W \cong W_\psi \natural W_F$. There is also a corresponding section $\sigma = \sigma(\psi) \in \text{Sec}(\pi)$ where $\text{Im}(\sigma) = j(H^4(W_f))$, j the natural homomorphism $H^4(W, M) \rightarrow H^4(M)$. By definition, see [5, Definition 2.50],

$$q_M^\sigma = \partial(R, -\lambda_R, \alpha_\psi),$$

and we define $f_\sigma: M \cong M(q_M^\sigma) \# M(F, d_\pi)$ to be the diffeomorphism on the boundary induced by the the splitting g_ψ . This shows that $\text{ASplit}(M) \rightarrow \text{Sec}(\pi)$ is onto.

Suppose that f_0 and f_1 are two splittings of M defining the same section σ . Then the H^* -equivalence class of f_i is determined by the almost diffeomorphism type of $M(q^{f_i})$. Now the Poincaré dual of $\text{Im}(\sigma)$ is a finitely generate free abelian group $\hat{F} \subset H_3(M)$. We choose a basis $\{x_1, \dots, x_b\}$ for \hat{F} and this is represented by a set of disjoint embeddings $\phi: \sqcup_{i=1}^b D^4 \times S^3 \subset M$. We let M_ϕ be the outcome of surgery on ϕ . Clearly there are choices ϕ_0 and ϕ_1 for ϕ so that

$M_{\phi_i} \cong M(q^{f_i})$. We claim that the almost diffeomorphism type of M_ϕ is independent of the choice of ϕ and this implies that $\sigma: \text{ASplit}(M) \rightarrow \text{Sec}(\pi)$ is injective.

To prove the claim, let W_ϕ be the trace of surgeries on ϕ and let $W_1 := W \cup_M W_\phi$ be the union of W_ϕ and our original 3-connected coboundary. By construction, we see that there is a fixed $\alpha_\sigma \in R^*$ such that the characteristic form of W_1 is isomorphic to $(R, \lambda_R, \alpha_\sigma) \oplus (H_1, \lambda_1, \alpha_1)$ where $(H_1, \lambda_1, \alpha_1)$ is nonsingular. It follows that the almost diffeomorphism type of M_ϕ is well-defined. \square

We conclude this subsection by identifying a simpler complete almost diffeomorphism invariant of 2-connected M . Recall that $\text{Aut}(b)$, the group of automorphisms of b , acts on $\mathcal{Q}(b)$, the set of refinements of b . We define the reduced refinement of M to be the function

$$\tilde{q}_M^\circ: S_2 \rightarrow \mathcal{Q}(b)/\text{Aut}(b), \quad h \mapsto [q^h].$$

For the following simple corollary to Theorem 3.5, recall that $\hat{S}_2 = \frac{d_\pi}{2} S_{d_\pi} \subseteq S_2$.

Corollary 3.6. *Let M_0 and M_1 have the same base (G, b, p) and reduced refinements \tilde{q}_0° and \tilde{q}_1° . The following are equivalent:*

- (i) M_0 is almost diffeomorphic to M_1 ;
- (ii) $\tilde{q}_0^\circ(\hat{S}_2) = \tilde{q}_1^\circ(\hat{S}_2)$;
- (iii) $\tilde{q}_0^\circ(\hat{S}_2) \cap \tilde{q}_1^\circ(\hat{S}_2) \neq \emptyset$.

\square

3.3. Inertia and reactivity in more detail. In order to proceed from the almost diffeomorphism classification of 2-connected 7-manifolds to the full diffeomorphism classification we need a better understanding of the inertia group of M

$$I(M) = \{\Sigma \mid M \sharp \Sigma \cong M\}.$$

We therefore begin with some general properties of the inertia group in dimension 7 and *do not assume that M is 2-connected*. Computing $I(M)$ is a delicate problem. An important and simpler first step is to compute the (integral) (co)homology inertia group of M ,

$$I_H(M) \subseteq I(M),$$

where we recall that a homotopy sphere $\Sigma \in I_H(M)$ if and only if there is a diffeomorphism $f: M \sharp \Sigma \cong M$ such that $H^*(f) = \text{Id}$, regarding $M \sharp \Sigma$ and M as the same topological space. The main result of this subsection, Proposition 3.10, establishes a lower bound on $I_H(M)$ for 2-connected M : see Remark 3.11.

To begin, let $\text{Diff}(M)$ be the group of diffeomorphisms of M . Recall that a pseudo-isotopy between diffeomorphisms $f_0, f_1: M \cong M$ is a diffeomorphism $F: M \times I \cong M \times I$ which restricts to f_i on $M \times \{i\}$. Pseudo-isotopy defines an equivalence relation on the group of diffeomorphisms of M and for future use we define

$$\tilde{\pi}_0 \text{Diff}(M) = \{[f] \mid f: M \cong M\}$$

to be the group of the pseudo-isotopy classes of spin diffeomorphisms of M . Here we remind the reader that M is spin and all the diffeomorphisms used to define $\pi_{\text{Diff}}(M)$ are spin diffeomorphisms.

We now consider the problem of when an almost diffeomorphism is pseudo-isotopic to a diffeomorphism. Let $f: M_0 \cong M_1$ be an almost diffeomorphism with singular set $\{m_1, \dots, m_a\}$. We now associate a diffeomorphism class of homotopy 7-sphere to each singular point m_i . Let $D_i^7 \ni m_i$ be a small disc containing m_i and no other singular points of f . The manifold $f(D_i^7) \subset M_1$ is a co-dimension zero submanifold of M_1 and so inherits a smooth structure from M_1 such that

$$\hat{f}_i := f|_{D_i^7 - \{m_i\}}: D_i^7 - \{m_i\} \cong f(D_i^7 - \{m_i\})$$

is a diffeomorphism. We can therefore define the smooth homotopy 7-sphere

$$\Sigma_i := D_i^7 \cup_{\hat{f}_i} (-f(D_i^7))$$

by gluing D_i^7 and $-f(D_i^7)$ together along \hat{f}_i . From the definition of pseudo-isotopy we deduce that the diffeomorphism class of the homotopy sphere

$$\Sigma_f := \Sigma_1 \sharp \Sigma_2 \sharp \dots \sharp \Sigma_a$$

is invariant under pseudo-isotopies. Moreover, it is clear that f defines a diffeomorphism

$$f: M \cong M \# \Sigma_f, \quad (33)$$

that f is pseudo-isotopic to a diffeomorphism if and only if $\Sigma_f \cong S^7$ and that $\Sigma_{f \circ g} = \Sigma_f \# \Sigma_g$. It follows that there is a *singularity homomorphism*

$$\partial: \tilde{\pi}_0 \text{ADiff}(M) \rightarrow \Theta_7 \quad [f] \mapsto \Sigma_f,$$

with kernel isomorphic to the image of $\tilde{\pi}_0 \text{Diff}(M)$ in $\tilde{\pi}_0 \text{ADiff}(M)$. If $\tilde{\pi}_0 \text{ADiff}_H(M) \subset \tilde{\pi}_0 \text{ADiff}(M)$ is the subgroup of pseudo-isotopy classes inducing the identity on $H^*(M)$, then we define the singularity homomorphism $\partial_H: \tilde{\pi}_0 \text{ADiff}_H(M) \rightarrow \Theta_7$ to be the restriction of ∂ . From (33) we see that

$$I_H(M) = \text{Im}(\partial_H) \quad \text{and} \quad I(M) = \text{Im}(\partial). \quad (34)$$

Given an almost diffeomorphism $f: M \cong M$ with singular points $\{m_1, \dots, m_a\}$, we now show how to determine $\Sigma_f \in \Theta_7$ using the mapping torus of f . The mapping torus of f is the almost smooth manifold constructed from the cylinder $M \times I$ by using f to identify points at either end:

$$T_f := (M \times [0, 1]) / (m, 0) \sim (f(m), 1).$$

Since f is an almost diffeomorphism, the closed 8-manifold T_f admits a smooth structure except perhaps at points $\{\overline{m}_1, \dots, \overline{m}_a\}$ corresponding to the singular points of f . Indeed if $B_i^8 \ni \overline{m}_i$ are small open balls containing precisely one singular point each of T_f then

$$W_f := T_f - \sqcup_{i=1}^a B_i^8 \quad (35)$$

is a compact smooth manifold with boundary

$$\partial W_f \cong \sqcup_{i=1}^a \Sigma_i.$$

We choose a spin structure on T_f and denote the corresponding 8-dimensional almost smooth spin manifold by T_f also: no confusion shall arise since we are interested only in the characteristic number

$$p^2(f) := \langle p_{T_f}^2, [T_f] \rangle \in \mathbb{Z},$$

which depends only on the oriented almost diffeomorphism type of T_f since $2p_{T_f} = p_1(T_f)$ and $H^8(T_f) \cong \mathbb{Z}$ (in fact p_{T_f} is independent of the choice of spin structure by [3, p. 170]). It follows that $p^2(f)$ is an invariant of the pseudo-isotopy class of f . For the statement of the next lemma, we recall the renormalised Eells-Kuiper invariant of a homotopy sphere Σ , $\mu(\Sigma) \in \mathbb{Z}/28$ defined in (3). By [12, (13)], $\mu(\Sigma_1) = \mu(\Sigma_2)$ if and only if $\Sigma_1 \cong \Sigma_2$.

Lemma 3.7. *For every almost diffeomorphism $f: M \cong M$ the following hold:*

- (i) $p^2(f) \in 8\mathbb{Z}$,
- (ii) $\mu(\Sigma_f) = \frac{p^2(f)}{8} \in \mathbb{Z}/28$,
- (iii) f is pseudo-isotopic to a diffeomorphism if and only if $p^2(f) \in 224\mathbb{Z}$.

Proof. (i) This follows since by Lemma 2.2(iii), p_{T_f} is characteristic for the intersection form of T_f . Hence by [30, Lemma 5.2, §5], $p^2(f) \equiv \sigma(T_f) \pmod{8}$. But by Novikov additivity, the signature of T_f is zero.

(ii) This follows since W_f defined in (35) above is a smooth spin coboundary for Σ_f and so can be used to compute $\mu(\Sigma_f)$. Since $\sigma(W_f) = \sigma(T_f) = 0$, applying (4) gives the result.

(iii) The almost diffeomorphism f is pseudo-isotopic to a diffeomorphism if and only if $\Sigma_f \cong S^7$. Hence (iii) follows directly from (ii). \square

In the light of Lemma 3.7, we define the function

$$p^2: \tilde{\pi}_0 \text{ADiff}(M) \rightarrow \mathbb{Z}, \quad [f] \mapsto p^2(f).$$

Since the image of p^2 plays a key role, we define non-negative integers called the *reactivity* of M , $R(M)$, and the *(co)homologically fixed reactivity* of M , $R_H(M)$, by the following equations

$$p^2(\text{ADiff}(M)) = R(M)\mathbb{Z} \quad \text{and} \quad p^2(\text{ADiff}_H(M)) = R_H(M)\mathbb{Z}.$$

By Lemma 3.7 (i), $R(M)$ and $R_H(M)$ are both divisible by 8. By Lemma 3.7 (ii) and the definition of reactivity we have

Proposition 3.8.

- (i) $I(M) = \frac{R(M)}{8}\Theta_7$,
- (ii) $I_H(M) = \frac{R_H(M)}{8}\Theta_7$.

□

For other problems, for example counting the number of deformation equivalence classes of G_2 -structures on M as in [6], it is important to know the value of p^2 for diffeomorphisms. Hence we define the *smooth reactivity* of M , $R^{\text{Diff}}(M)$, and the *smooth (co)homologically fixed reactivity* of M , $R_H^{\text{Diff}}(M)$ by the equations

$$p^2(\text{Diff}(M)) = R^{\text{Diff}}(M)\mathbb{Z} \quad \text{and} \quad p^2(\text{Diff}_H(M)) = R_H^{\text{Diff}}(M)\mathbb{Z}.$$

By Lemma 3.7 (iii) we have

Lemma 3.9.

- (i) $R^{\text{Diff}}(M) = \text{lcm}(R(M), 224)$,
- (ii) $R_H^{\text{Diff}}(M) = \text{lcm}(R_H(M), 224)$.

□

We now turn to the construction of almost diffeomorphisms $f \in \text{ADiff}(M)$ on 2-connected M with $p^2(f) \neq 0$. Recall that d_π is the divisibility of $\pi(p_M) \in H^4(M)/TH^4(M)$ and that $\tilde{d}_\pi = \text{lcm}(4, d_\pi)$.

Proposition 3.10. *If M is 2-connected then $R_H(M) \mid 2\tilde{d}_\pi$; i.e. $2\tilde{d}_\pi\mathbb{Z} \subseteq p^2(\text{ADiff}_H(M))$.*

Remark 3.11. If M is 2-connected, Propositions 3.8 and 3.10 together give $\frac{\tilde{d}_\pi}{4}\Theta_7 \subseteq I_H(M)$. In Corollary 4.17 (ii) below we will show that $R_H(M) = 2\tilde{d}_\pi$ and hence $\frac{\tilde{d}_\pi}{4}\Theta_7 = I_H(M)$.

For the proof of Proposition 3.10 it will be useful to compute the characteristic number $p^2(T_f)$ using a co-bounding spin 8-manifold W . We define the closed almost smooth 8-manifold

$$X_f := W \cup_f -W.$$

Lemma 3.12. *With the notation above, $p^2(f) = \langle p_{X_f}^2, [X_f] \rangle$.*

Proof. Since $p^2(f) = \langle p_{T_f}^2, [T_f] \rangle$ and $\langle p_{X_f}^2, [X_f] \rangle$ are characteristic numbers, it suffices to prove that T_f is oriented bordant to X_f . Consider the manifolds $M \times I$ and $W \sqcup -W$. Both have boundary $M \sqcup -M$, T_f is formed from $M \times I$ by gluing M to $-M$ via f and X_f if formed from $W \sqcup -W$ by gluing M to $-M$ via f . It therefore suffices to prove that $W \sqcup -W$ is bordant relative to the boundary to $M \times I$. But the manifold $W \times I$ is a rel. boundary bordism from $W \sqcup -W$ to $M \times I$, and we are done. □

Proof of Proposition 3.10. We assume $d_\pi \neq 0$, since otherwise there is nothing to prove. By Theorem 3.5 or by [38, Theorem 1], we may decompose M as a connected sum of spin manifolds

$$M \cong M_1 \# M_2$$

where $M_1 = M(\mathbb{Z}, d_\pi) = S^3 \tilde{\times}_{d_\pi} S^4$ is the total space of a 3-sphere bundle over S^4 defined in (32). We shall produce the required almost diffeomorphisms on the manifold M_1 and then extend by the identity to M . Let

$$M_1^\bullet := M_1 - \text{Int}(D^7)$$

be M_1 minus a small open disc. Since M_1 is the total space of an S^3 -bundle over S^4 there is a diffeomorphism

$$M_1^\bullet \cong (D^3 \tilde{\times}_{d_\pi} S^4) \cup_{S^2 \times D^4} (D^3 \times D^4)$$

where $D^3 \tilde{\times}_{d_\pi} S^4$ is a tubular neighbourhood of a section of $M_1 \rightarrow S^4$ and $D^3 \times D^4$ is a 3-handle added to $D^3 \tilde{\times}_{d_\pi} S^4$ along the tubular neighbourhood of a fibre 2-sphere, $S^2 \times D^4 \subset S^2 \tilde{\times}_{d_\pi} S^4$.

By [34, p. 171 (2)] we may identify $\pi_3(SO(4))$ as the group of pairs of integers (n, p) where $n \equiv p \pmod{2}$, so that the corresponding S^3 -bundle over S^4 has Euler class $n \in H^4(S^4) = \mathbb{Z}$ and first

Pontrjagin class $2p$. Let $\gamma_{n,p}: (D^3, S^2) \rightarrow (SO(4), \text{Id})$ be a smooth function representing (n, p) . We define a diffeomorphism

$$f_{n,p}^\bullet: M_1^\bullet \cong M_1^\bullet$$

where $f_{n,p}^\bullet|_{D^3 \times_{\alpha} S^4}$ is the identity and on the 3-handle we use the D^3 co-ordinate to twist the D^4 -coordinate using $\gamma_{n,p}$. To be explicit:

$$f_{n,p}^\bullet|_{D^3 \times D^4}(u, v) = (u, \gamma_{n,p}(u)(v)).$$

We observe that there is a copy of $S^3 \vee S^4 \subset M_1^\bullet$ the restriction $f_{n,p}^\bullet|_{S^3 \vee S^4}$ is the identity and such that M_1^\bullet deformation retracts to $S^3 \vee S^4$. It follows that $f_{n,p}^\bullet$ acts trivially on cohomology.

By the Alexander trick, we may extend $f_{n,p}^\bullet$ to an almost diffeomorphism of M_1 and we denote any choice of extension by $f_{n,p}$. Since $f_{n,p}^\bullet$ acts trivially on cohomology, so does $f_{n,p}$. Since M_1 admits a unique spin structure for each orientation and since $f_{n,p}$ is orientation preserving, $f_{n,p}$ is a spin almost diffeomorphism. Up to pseudo-isotopy, we may assume that $f_{n,p}$ is the identity on a disc and hence we may extend $f_{n,p}$ to M by taking the connected sum with the identity on M_2 . Thus we define the almost diffeomorphism

$$g_{n,p} := f_{n,p} \# \text{Id}_{M_2}: M \cong M,$$

and note that $g_{n,p}$ acts trivially on cohomology and is a spin almost diffeomorphism. We claim that

$$p^2(g_{n,p}) = p^2(f_{n,p}) = d_\pi(2p - nd_\pi). \quad (36)$$

The manifold $M_1 \cong S^3 \tilde{\times}_{d_\pi} S^4$ bounds the 8-dimensional D^4 -bundle $W_1 := D^4 \tilde{\times}_{d_\pi} S^4$, and we let W_2 be any spin coboundary for M_2 . We form the twisted doubles $X_{f_{n,p}} := W_1 \cup_{f_{n,p}} (-W_1)$ and

$$X_{g_{n,p}} := (W_1 \natural W_2) \cup_{g_{n,p}} (-W_1 \natural -W_2) \cong X_{f_{n,p}} \# (W_2 \cup_{\text{id}} (-W_2)). \quad (37)$$

Applying Lemma 3.12 we have,

$$p^2(g_{n,p}) = \langle p^2(X_{g_{n,p}}), [X_{g_{n,p}}] \rangle = \langle p^2(X_{f_{n,p}}), [X_{f_{n,p}}] \rangle = p^2(f_{n,p}),$$

where the second equality holds by (37) since the characteristic number p^2 is a bordism invariant which is additive for connected sums and $W_2 \cup (-W_2) = \partial(W_2 \times I)$. Writing $X_{n,p} := X_{f_{n,p}}$, it therefore remains to compute $\langle p^2(X_{n,p}), [X_{n,p}] \rangle$. From the construction of $X_{n,p}$ we see that $H_4(X_{n,p}) \cong \mathbb{Z}(x) \oplus \mathbb{Z}(y)$ where x is represented by the zero section of W_1 and $y = [D^4 \cup D^4]$ is represented by an embedded 4-sphere obtained by gluing two fibres of the D^4 -bundle W_1 together, one from each copy of W_0 . By construction, the normal bundle of the 4-sphere $D^4 \cup D^4$ has characteristic function $\gamma_{n,p}$ and hence Euler number n . It follows that the intersection form of $X_{n,p}$ with respect to the basis $\{x, y\}$ is given by the following matrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix}$$

Moreover since x is represented by an embedded 4-sphere with tubular neighbourhood diffeomorphic to $D^4 \tilde{\times}_{d_\pi} S^4$ and since y is represented by an embedded 4-sphere with normal bundle $\gamma_{n,p}$, we have $p_{X_{n,p}}(x) = d_\pi$ and $p_{X_{n,p}}(y) = p$. We conclude that the Poincaré dual of $p_{X_{n,p}}$ is given by

$$PD(p_{X_{n,p}}) = (p - nd_\pi)x + d_\pi y.$$

It follows that $\langle p^2_{X_{n,p}}, [X_{n,p}] \rangle = 2d_\pi(p - nd_\pi) + nd_\pi^2 = d_\pi(2p - nd_\pi)$, and the claim (36) is proven.

Finally we need to choose n and p so that $d_\pi(2p - nd_\pi) = 2\tilde{d}_\pi$. Recall that we may choose n and p freely subject to the constraint that $n \equiv p \pmod{2}$. If $d_\pi = 4k + 2$, then $2\tilde{d}_\pi = 4d_\pi$ and we choose $(n, p) = (0, 2)$. If $d_\pi = 4k$, then $2\tilde{d}_\pi = 2d_\pi$ and we set $(n, p) = (1, 2k + 1)$. \square

3.4. The proof of the main classification theorem. The mod 28 distillation of M is the quadruple $(H^4(M), q_M^\circ, \mu_M, p_M)$ where q_M° is the quadratic linking family of M as in Definition 2.15 and the generalised Eells-Kuiper invariant

$$\mu_M: S_{d_\pi} \rightarrow \mathbb{Q}/\hat{d}_\pi \mathbb{Z}$$

is the mod 28 Gauss refinement of q_M° defined by (26). In this subsection we prove Theorem 1.3 which states that mod 28 distillations give a complete invariant of diffeomorphisms of 2-connected M . For the remainder of the subsection M is 2-connected.

Recall that the (renormalised) classical Eells-Kuiper invariant, as defined by (3), gives a group isomorphism

$$\Theta_7 \cong \mathbb{Z}/28\mathbb{Z}, \quad \Sigma \mapsto \mu(\Sigma) := \mu_\Sigma(0).$$

The following lemma is obvious from the definitions of q_M° and μ_M .

Lemma 3.13. *For all $\Sigma \in \Theta_7$, $q_{M\sharp\Sigma}^\circ = q_M^\circ$ and $\mu_{M\sharp\Sigma} = \mu_M + [\mu(\Sigma)]$, where $[\mu(\Sigma)]$ is the mod \hat{d}_π reduction of $\mu(\Sigma)$.* \square

Proof of Theorem 1.3. The existence of a smooth M with mod 28 distillation isomorphic to a prescribed $(G, q^\circ, \mu, p) \in \mathcal{Q}_\mu$ follows from the corresponding existence statement in Theorem 1.2, since Lemma 3.13 lets us freely change the Eells-Kuiper invariant of a manifold with a prescribed refinement $(G, q^\circ, p) \in \mathcal{Q}_q$.

By Lemma 1.7, which is proven in (24), the generalised Eells-Kuiper invariant is a diffeomorphism invariant. Now we suppose $F^\#(q_{M_0}^\circ, \mu_{M_0}, p_{M_0}) = (q_{M_1}^\circ, \mu_{M_1}, p_{M_1})$. As explained in (33), Theorem 1.2 means that there is a homotopy sphere Σ and a diffeomorphism

$$f: M_0\sharp\Sigma \cong M_1$$

such that $H^*(f) = F: H^4(M_1) \cong H^4(M_0)$. It remains to show that $\Sigma \in I_H(M_0)$. For if so, there is a diffeomorphism

$$h: M_0 \cong M_0\sharp\Sigma$$

with $H^*(h) = \text{Id}$ and then $f \circ h: M_0 \cong M_1$ is a diffeomorphism with $H^*(f \circ h) = H^*(f) = F$.

Since f is a diffeomorphism it preserves the mod 28 Gauss refinements. Applying Lemma 3.13 we have

$$\mu_{M_1} = F^\#(\mu_{M_0} + \mu(\Sigma)) = F^\#(\mu_{M_0}) + \mu(\Sigma).$$

On the other hand, our assumption is that $F^\#(\mu_{M_0}) = \mu_{M_1}$. Since $d_{M_0} = d_{M_1}$,

$$\mu(\Sigma) = \mu_{M_1} - F^\#(\mu_{M_0}) = 0 \in \mathbb{Z}/d_{M_1}\mathbb{Z} = \mathbb{Z}/d_{M_0}\mathbb{Z}.$$

By Remark 3.11, $\Sigma \in I_H(M_0)$ and this completes the proof. \square

Proof of Theorem 1.5. That the functor $\mathcal{Q}: \mathcal{M}_{7,2}^{spin} \rightarrow \mathcal{Q}_\mu$ is surjective and faithful is a restatement of Theorem 1.3. To see that \mathcal{Q} is additive and compatible with orientation reversal, let $i = 0, 1$, and let $M_i = \partial W_i$ where W_i has characteristic form $(H_i, \lambda_i, \alpha_i)$ with boundary refinement q_i° . The mod 28 Gauss refinement of M_i is $(q_i^\circ, \partial g_{W_i})$ where we set $\partial g_{W_i} := g_{W_i} \bmod \hat{d}_\pi$ as in (26). Since the characteristic form of $-W_i$ is $(H_i, -\lambda_i, \alpha_i)$ and the characteristic form of the boundary connected sum $W_0\sharp W_1$ is $(H_0, \lambda_0, \alpha_0) \oplus (H_1, \lambda_1, \alpha_1)$, it follows that mod 28 Gauss refinements of $-M_i$ and $M_0\sharp M_1$ are $(-q_i^\circ, -\partial g_{W_i})$ and $(q_1^\circ, \partial g_{W_0}) \oplus (q_1^\circ, \partial g_{W_1})$ respectively. \square

3.5. Smooth splitting functions. In this subsection we consider connected sum splittings of 2-connected M in the smooth category and we prove a smooth analogue of Theorem 3.5. We also prove Theorem 1.4 which is the smooth analogue of Corollary 3.6. Throughout the subsection M is 2-connected.

Let (T, b) be a torsion form and let d be an even integer. We define the set

$$\hat{\mathcal{Q}}_d(b) := \{(q, s)\} \subset \mathcal{Q}(b) \times \mathbb{Q}/d\mathbb{Z},$$

which consists of pairs of quadratic refinements q of b and rational residues mod d where $A(q) \equiv s \bmod \mathbb{Z}$. By Theorem 1.3, a rational homotopy sphere M with linking form isomorphic to (T, b) is

classified up to diffeomorphism by the pair $(q_M, \mu(M)) \in \hat{\mathcal{Q}}_{28}(b)$. We denote this rational homotopy sphere by

$$M = M(q, s),$$

where $s = \mu(M)$. Suppose that we are given a base (G, b, p) with $F = G/T \cong \mathbb{Z}^b$ and $\pi(p) \in F$ of divisibility d_π . By Theorem 3.5, if M has base $(H^4(M), p_M, b_M) \cong (G, b, p)$, there is a connected sum splitting

$$f: M \cong M(q^f, s(f)) \# M(F, d_\pi),$$

for some rational homotopy sphere $M(q^f, s(f))$. We recall that $\sigma(f) \in \text{Sec}(\pi)$ is the section defined by $\text{Im}(\sigma) = f^*(H^4(M(F, d_\pi)))$. In considering the uniqueness of $M(q, s)$ in such a splitting, we note that by Theorem 1.3 $I(M(F, d_\pi)) = \hat{d}_\pi \Theta_7$ (see also Remark 3.11). As a consequence, we see that if $(q_i, s_i) \in \mathcal{Q}_{28}(b)$, $i = 0, 1$, have $s_0 \equiv s_1 \pmod{\hat{d}_\pi \mathbb{Z}}$, then there is a diffeomorphism

$$g: M(q, s_0) \# M(F, d_\pi) \cong M(q, s_1) \# M(F, d_\pi) \quad (38)$$

such that $H^*(g)$ preserves the induced splittings of H^4 . Hence we define two splittings f_0 and f_1 to be H^* -equivalent if there is a diffeomorphism $g: M \cong M$ with $H^*(g) = \text{Id}$ and if there are almost diffeomorphisms $g_T: M(q_0) \cong M(q_1)$ and $g_F: M(F, d_\pi) \cong M(F, d_\pi)$ with $\Sigma_{g_T} = -\Sigma_{g_F} \in \hat{d}_\pi \Theta_7$ such that the following diagram commutes up to pseudo-isotopy:

$$\begin{array}{ccc} M & \xrightarrow{f_0} & M(q^{f_0}, s(f_0)) \# M(F, d_\pi) \\ \downarrow g & & \downarrow g_T \# g_F \\ M & \xrightarrow{f_1} & M(q^{f_1}, s(f_1)) \# M(F, d_\pi) \end{array}$$

We define $\text{Split}(M) := \{[f]\}$ to be the set of H^* -equivalence classes of splittings of M . We also define the *smooth splitting function* of M

$$\hat{q}_M^\bullet: \text{Sec}(\pi) \rightarrow \hat{\mathcal{Q}}_{\hat{d}_\pi}(b), \quad \sigma \mapsto \hat{q}_M^\sigma := (q_M^{k(\sigma)}, \mu_M(k(\sigma))),$$

where we recall that $k(\sigma) \in S_{d_\pi}$ is defined by $k(\sigma) \in S_{d_\pi} \cap \text{Im}(\sigma)$. From the diffeomorphism in (38), we see that $M(q^{k(\sigma)}, \mu(k(\sigma))) \# M(F, d_\pi)$ gives a well-defined diffeomorphism type for each section σ .

Theorem 3.14. *Let M have smooth splitting function $\hat{q}_M^\bullet: \text{Sec}(\pi) \rightarrow \hat{\mathcal{Q}}_{\hat{d}_\pi}(b)$. For each $\sigma \in \text{Sec}(\pi)$ there is a unique H^* -equivalence class of splitting*

$$f_\sigma: M \cong M(\hat{q}_M^\sigma) \# M(F, d_\pi).$$

Consequently the map $\text{Split}(M) \rightarrow \text{Sec}(\pi), [f] \mapsto \sigma(f)$ is a bijection.

Proof. The proof is a refined version of the proof Theorem 3.5 and we adopt the notation of that proof so that M has 3-connected coboundary W . Specifically, the proof of the existence of f_σ is verbally the same, except that now by [5, Definition 2.50] we have

$$\hat{q}_M^\sigma = \left(\partial(R, -\lambda_R, \alpha_\psi), \frac{\lambda_R^{-1}(\alpha_\psi, \alpha_\psi) - \sigma(\lambda_R)}{8} \right).$$

Hence the splitting $W \cong W_\psi \natural W_F$ defines the splitting $f_\sigma: M \cong M(\hat{q}_M^\sigma) \# M(F, d_\pi)$.

To show that splittings f_0 and f_1 defining the same section σ are H^* -equivalent, we consider the nonsingular characteristic form $(H_1, \lambda_1, \alpha_1)$. The symmetric form (H_1, λ_1) has a Lagrangian $L \subset H_1$ corresponding to $H^3(M)$ and hence $\alpha_1(L) = d_\pi \mathbb{Z}$. The proof of Proposition 3.10 now shows that $\frac{\hat{d}_\pi}{4}$ divides $(\lambda_1^{-1}(\alpha_1, \alpha_1) - \sigma(\lambda_1))/8$. Consequently the diffeomorphism type of M_ϕ is determined up to connected sum with $\Sigma \in \hat{d}_\pi \Theta_7$, and this shows that f_0 and f_1 are H^* -equivalent splittings. \square

Proof of Theorem 1.4. Let $\text{Aut}(b)$ act on $\hat{\mathcal{Q}}_{\hat{d}_\pi}(b)$ by $F \cdot (q, s) = (q \circ F, s)$ and let $[q, s]$ denote the $\text{Aut}(b)$ equivalence class of (q, s) . We define the map

$$\beta: \hat{\mathcal{Q}}_{\hat{d}_\pi}(b) \rightarrow 2T/\text{Aut}(b) \times \mathbb{Q}/\hat{d}_\pi \mathbb{Z}, \quad (q, s) \mapsto ([\beta_q], s).$$

Since $A(q) = s$, it follows from Theorem 2.11 that $[q_0, s_0] = [q_1, s_1]$ if and only if $\beta(q_0, s_0) = \beta(q_1, s_1)$.

Now let M_0 and M_1 have smooth splitting functions $\hat{q}_i^\bullet : \text{Sec}(\pi) \rightarrow \hat{\mathcal{Q}}_{\hat{d}_\pi}(b)$, $i = 0, 1$. By definition, these splitting functions descend to splitting functions

$$\hat{q}_i^\circ : S_{d_\pi} \rightarrow \hat{\mathcal{Q}}_{\hat{d}_\pi}(b).$$

By Theorem 3.14, M_0 and M_1 are diffeomorphic if and only if there are sections σ and σ' , a homotopy sphere $\Sigma \in \hat{d}_\pi \Theta$ and a diffeomorphism $M(\hat{q}_0^\sigma) \cong M(\hat{q}_1^{\sigma'}) \# \Sigma$. But this happens if and only if there is an isomorphism $F : q_0^\sigma \cong q_1^{\sigma'}$ and $\mu_0(k(\sigma)) = \mu_1(k(\sigma'))$; *i.e.* if and only if $\beta(\hat{q}_0^\sigma) = \beta(\hat{q}_1^{\sigma'})$. The reduced splitting function of M_i , defined in the introduction just prior to the statement of Theorem 1.4, is the function,

$$\overline{q}_i^\circ : S_{d_\pi} \rightarrow (2T/\text{Aut}(b)) \times \mathbb{Q}/\hat{d}_\pi \mathbb{Z}, \quad k \mapsto \beta(\hat{q}_i^k).$$

The above shows that M_0 and M_1 are diffeomorphic if and only if the images of \overline{q}_0° and \overline{q}_1° intersect and consequently this happens if and only if the images of \overline{q}_0° and \overline{q}_1° coincide. \square

4. AUTOMORPHISMS OF $H^4(M)$

The smooth classification Theorem 1.3 implies that the number of different smooth structures on the same 2-connected almost-smooth 7-manifold corresponds to the number of different mod 28 Gauss refinements of the linking family $(H^4(M), q_M^\circ, p_M)$. The first estimate of the number of smooth structures on M , $\hat{d}_\pi = \gcd(\frac{d_\pi}{4}, 28)$, only counts smooth structures on M modulo almost diffeomorphisms that act trivially on $H^4(M)$. To get the full picture, we need to understand how automorphisms of the quadratic linking family q_M° act on the Gauss refinements. We begin this process in Section 4.2.

Conveniently enough, it turns out that this problem can be reduced to understanding how automorphisms of the base $(H^4(M), b_M, p_M)$ act on linked function: see Proposition 4.10 in Section 4.3. While we do not have a complete description of this action in general we still have control up to a factor 2^r where $r \in 0, 1, 2$ is explicitly defined in (42). Moreover it is feasible to understand it for explicit examples: see Examples 4.7, 4.8, 4.12 and 4.13. With Proposition 4.10 in hand, we proceed in Section 4.4 to determine the reactivity of 2-connected M in terms of r and the integer d_κ defined in Section 4.1.

4.1. Notation. We begin by setting up some terminology. Given a finitely generated abelian group G , $p \in 2G$, and $b : T \times T \rightarrow \mathbb{Q}/\mathbb{Z}$ a torsion form, let Aut_b denote the group of isomorphisms $F : G \rightarrow G$ preserving p and b . If q° is a family of quadratic refinements of (G, b, p) , let Aut_{q° be the subgroup of Aut_b that preserves q too.

Let $\pi : G \rightarrow G/T$ be the projection to the free quotient of G . Let $\text{Shr}_p \subseteq \text{Aut}_b$ be the subgroup of “pure shears”, *i.e.* F acting trivially on T and G/T . In other words, $F = \text{Id}_G + \rho \circ \pi$ for some homomorphism $\rho : G/T \rightarrow T$ such that $\rho(\frac{\pi(p)}{d_\pi})$ is d_π -torsion (the last condition is equivalent to $F(p) = p$); so actually Shr_p does not depend on b at all. Similarly let $\text{Shr}_{p/2} = \text{Shr}_p \cap \text{Aut}_{q^\circ}$, the subgroup of shears in Aut_{q° . For $h \in S_2$, $(F^\# q)^h = q^{F(h)} = q^{h+\rho(\frac{\pi(p)}{2})} = q_{-\rho(\frac{\pi(p)}{2})}^h$, so for F to preserve q we need $\rho(\frac{\pi(p)}{2}) = 0$. Hence $\text{Shr}_{p/2}$ simply corresponds to homomorphisms $\rho : G/T \rightarrow T$ such that $\rho(\frac{\pi(p)}{d_\pi})$ is $\frac{d_\pi}{2}$ -torsion. In particular: $\text{Shr}_{p/2}$ actually depends on neither q° nor b but only on p , and if $F \in \text{Shr}_p$ then $F^2 \in \text{Shr}_{p/2}$.

We say that a cyclic subgroup $C \subseteq T$ is a *split summand* if T is a direct sum of C and its b -orthogonal complement. We call $x \in T$ *split* if it generates a split summand; this is equivalent to

$$b(x, x) \not\equiv 0 \pmod{\frac{1}{n}},$$

where n is the order of x . (Then x cannot be divisible by any integer dividing n .)

Given the element $p \in G$ we consider the following notions of its divisibility (if p is a torsion element we set all three integers to be 0):

$$\begin{aligned} d &:= \text{Max}\{s \in \mathbb{Z} \mid r \text{ divides } p \in G\}, \\ d_\pi &:= \text{Max}\{s \in \mathbb{Z} \mid r \text{ divides } \pi(p) \in G/T\}, \\ d_o &:= \text{Max}\{s \mid s, m \in \mathbb{Z}, rm^2 \text{ divides } mp \in G\}. \end{aligned}$$

We have an obvious chain of divisibilities

$$2 \mid d \mid d_o \mid d_\pi.$$

Further $d = d_\pi$ if and only if $d_o = d_\pi$, since the latter implies that the maximum in the definition of d_o is attained with $m = 1$.

For an integer s , let $\text{ord}_2 s$ be the exponent of 2 in the prime factorisation of s ; e.g. $\text{ord}_2 2^j = j$.

Definition 4.1. A non-negative integer e is a *2-extremal exponent* for (G, p) if for some m such that $d_o m^2$ divides mp , $\text{ord}_2 m = e$.

Example 4.2. Let $p = (2^a, 2^c) \in \mathbb{Z} \times \mathbb{Z}/2^b\mathbb{Z}$ with $a, b \geq c \geq 1$. Then $d_\pi = 2^a$, $d_o = \max(2^c, 2^{a-b+c})$ and $d = 2^c$. The 2-extremal exponents are 0 for $a \leq b$, and $b - c$ for $a \geq b$.

4.2. The action of Aut_b on linked functions. Given $F \in \text{Aut}_b$ and any $k \in S_{d_\pi}$, set $t := F(k) - k \in T$ (not necessarily d_π -torsion, unless $F|_T$ is the identity) and $\beta_k := p - d_\pi k$, and let

$$P(F) := d_\pi^2 b(t, t) - 2d_\pi b(\beta_k, t) \in \mathbb{Q}/2d_\pi\mathbb{Z}. \quad (39)$$

In other words, $P(F) = \Delta(k, t)$ from (16). Equivalently, we can define $P(F)$ by

$$F^\# g = g + \frac{P(F)}{8} \pmod{\frac{d_\pi}{4}\mathbb{Z}} \quad (40)$$

for any linked function g (use that $(F^\# g)(k) = g(F(k)) = g(k + t) = g(k) + \frac{\Delta(k, t)}{8}$ by the condition (15) for g to be a linked function.) The first definition, (39), is independent of g and the second, (40), of k , so in fact P depends on neither. If F preserves a family of quadratic refinements, then taking g to be a Gauss refinement of that family shows that P takes values in $8\mathbb{Z}/2d_\pi\mathbb{Z}$ (in the next subsection we study a corresponding $8\mathbb{Z}/2\tilde{d}_\pi\mathbb{Z}$ -valued function \tilde{P}). Even if F does not preserve a family of quadratic refinements, the fact that the Arf invariant $\text{mod } \frac{1}{4}\mathbb{Z}$ of a quadratic refinement of (b, p) depends only on (b, p) itself shows that P takes values in $2\mathbb{Z}$. It is also clear from (40), or from (39) together with (19), that P is a homomorphism $\text{Aut}_b \rightarrow 2\mathbb{Z}/2d_\pi\mathbb{Z}$.

Let $j_\pi = \text{ord}_2 d_\pi$, and $j_o = \text{ord}_2 d_o$.

Lemma 4.3. $P(\text{Aut}_b) \subseteq d_o\mathbb{Z}/2d_\pi\mathbb{Z}$. If b lacks a split 2^{e+j_o} summand for some 2-extremal exponent e , then $P(\text{Aut}_b) \subseteq 2d_o\mathbb{Z}/2d_\pi\mathbb{Z}$.

Proof. Pick some $y \in G$ such that $m^2 d_o y = mp$. Then $s := Fy - y$ is an $m^2 d_o$ -torsion element. It suffices to show that

$$P(F) = m^2 d_o^2 b(s, s) \pmod{2d_o}, \quad (41)$$

because the RHS is d_o if the 2-primary part of s is split, and 0 otherwise.

Note that $u := \frac{d_\pi}{d_o}$ and $\frac{m}{u}$ are integers. Let $k := \frac{m}{u}y$. Then $k \in S_{d_\pi}$, and $ut = ms$, so (39) implies

$$P(F) = u^2 d_o^2 b(t, t) - 2ud_o b(\beta_k, t) = d_o^2 (ms, ms) - 2d_o (\beta_k, ms) \pmod{2d_o}.$$

Since $\beta_k = p - d_\pi k$ is m -torsion, (41) and the result follows. \square

If $F \in \text{Shr}_p \subseteq \text{Aut}_b$, i.e. $F = \text{Id}_G + \rho \circ \pi$ for some homomorphism $\rho : G/T \rightarrow T$, then $t = F(k) - k = \rho(\frac{\pi(p)}{d_\pi})$ independent of the choice of $k \in S_{d_\pi}$. Since $\frac{\pi(p)}{d_\pi} \in G/T$ is a primitive element of a free abelian group, we can prescribe its image under a homomorphism ρ arbitrarily. Determining the image $P(\text{Shr}_p)$ therefore amounts to computing the RHS of (39) for all d_π torsion elements $t \in T$.

Lemma 4.4. $4d_o\mathbb{Z}/2d_\pi\mathbb{Z} \subseteq P(\text{Shr}_p)$. If $j_\pi \neq j_o + 1$ or if b has no split 2^{j_π} summand, then $2d_o\mathbb{Z}/2d_\pi\mathbb{Z} \subseteq P(\text{Shr}_p)$.

Proof. The key claim is that there exists a d_π -torsion element t such that $b(\beta_k, t) = \frac{1}{u}$, where $d_\pi = ud_o$. By the non-degeneracy of b , this is equivalent to β_k having order at least u , and not being divisible by more than d_o . That any divisor of β_k also divides d_o is obvious, and if $m\beta_k = 0$ then mp is divisible by md_π , which indeed implies $u \mid m$ by the definition of d_o .

Let ρ be any homomorphism $G/T \rightarrow T$ mapping $\frac{\pi(p)}{d_\pi} \mapsto t$, and $F := \text{Id}_G + \rho \circ \pi \in \text{Shr}_p$. If the 2-primary part of t does not generate a split 2^{j_π} summand then $d_\pi^2 b(t, t)$ is divisible by $2d_\pi$, so

$$P(F) = d_\pi^2 b(t, t) - 2d_\pi b(\beta_k, t) = 2d_o \pmod{2d_\pi},$$

and we are done. Otherwise $P(F) = d_\pi - 2d_o = (u - 2)d_o \pmod{2d_\pi}$. The subgroup this generates is precisely $nd_o\mathbb{Z}/2d_\pi\mathbb{Z}$, where $n = \gcd(u-2, 2u) = \gcd(u-2, 4)$. Clearly n is 1 or 2 except when $j_\pi = j_o + 1$, in which case $n = 4$. \square

Lemmas 4.3 and 4.4 imply that the following is well-defined.

Definition 4.5. Define $r(G, p, b) \in \{0, 1, 2\}$ by

$$\text{Im } P = 2^r d_o \mathbb{Z}/2d_\pi \mathbb{Z}. \quad (42)$$

Remark 4.6. Lemmas 4.3 and 4.4 provide necessary conditions for $r = 0$ or $r = 2$. In particular, if G has no 2-torsion then $r = 1$. The next examples show that there are bases with $r = 0$ and bases with $r = 2$.

Example 4.7. Let $G = \mathbb{Z} \oplus \mathbb{Z}/2^j$, $b = \langle \frac{1}{2^j} \rangle$ and $p = (2^j, 0)$ (so $d_\pi = d_o = 2^j$). Then the shear $F : (x, y) \mapsto (x, x + y)$ has $P(F) = 2^j \pmod{2^{j+1}}$, i.e. $P(F) = d_o \pmod{2d_\pi}$. Thus $r = 0$.

Example 4.8. Let $G = \mathbb{Z} \oplus \mathbb{Z}/2^j$, $b = \langle \frac{1}{2^j} \rangle$ and $p = (2^j, 2^{j-1})$ (so $d_\pi = 2^j$, while $d_o = 2^{j-1}$). Now any $t \in T$ has $d_\pi^2 b(t, t) + 2d_\pi b(\beta_k, t) = 0 \pmod{2^{j+1}}$, so $r = 2$.

4.3. The action of Aut_{q° on Gauss refinements. Now let q° be a family of quadratic refinements of the base (G, b, p) , and let Aut_{q° denote its group of automorphisms. For an automorphism $F \in \text{Aut}_{q^\circ}$ we define $\tilde{P}(F) \in 8\mathbb{Z}/2\tilde{d}_\pi\mathbb{Z}$ by

$$\begin{aligned} \tilde{P}(F) &= P(F) \pmod{2d_\pi}, \\ \tilde{P}(F) &= 0 \pmod{8}. \end{aligned}$$

Equivalently, $\tilde{P}(F) = \tilde{\Delta}(k, t)$ for any $k \in S_{d_\pi}$ and $t := F(k) - k$. Now \tilde{P} is a homomorphism $\tilde{P} : \text{Aut}_{q^\circ} \rightarrow 8\mathbb{Z}/2\tilde{d}_\pi\mathbb{Z}$, such that

$$F^\# g = g + \frac{\tilde{P}(F)}{8} \pmod{\frac{\tilde{d}_\pi}{4}\mathbb{Z}} \quad (43)$$

for any Gauss refinement g of q° .

We can get some control on the image of the shear subgroup $\text{Shr}_{p/2} \subseteq \text{Aut}_{q^\circ}$ just from the observation that $F^2 \in \text{Shr}_{p/2}$ for any $F \in \text{Shr}_p$.

Lemma 4.9. $\tilde{P}(\text{Shr}_{p/2}) \supseteq 4d_o\mathbb{Z}/2\tilde{d}_\pi\mathbb{Z}$.

Proof. The proof of Lemma 4.4 showed that we can achieve $P(F) = 2d_o$ or $P(F) = 2d_o + d_\pi$ for some $F \in \text{Shr}_p$. Then $F^2 \in \text{Shr}_{p/2}$ has $P(F^2) = 4d_o$. \square

Proposition 4.10. $\text{Im } \tilde{P} = \{n \in 8\mathbb{Z}/2\tilde{d}_\pi\mathbb{Z} : n \pmod{2d_\pi} \in \text{Im } P\} = \text{lcm}(8, 2^r d_o)\mathbb{Z}/2\tilde{d}_\pi\mathbb{Z}$.

Proof. If d_π is not divisible by 4 then the result follows from Lemma 4.9. If $4 \mid d_\pi$ and $F \in \text{Aut}_b$, then $P(F) = n \in 8\mathbb{Z}/2d_\pi\mathbb{Z}$ implies that for any $k \in S_{d_\pi}$, $t := F(k) - k$ and $h := \frac{d_\pi}{2}k$, we get $q^h(\frac{d_\pi}{2}t) = 0$. Thus $(F^\# q)^h$ and q^h , which have equal inhomogeneity $\beta_h = p - 2h$ by definition, also have equal Arf invariant by (9). Therefore by Theorem 2.11 there is an automorphism F_T of (T, b) such that $(F^\# q)^h \circ F_T = q^h$ (necessarily F_T fixes β_h).

Now suppose that σ is a section of π , and $k \in \text{Im } \sigma \cap S_{d_\pi}$ (cf. Remark 2.19). Then $G \cong \text{Im } \sigma \oplus T$, and we may define $\text{Id}_{\text{Im } \sigma} + F_T \in \text{Aut}_b$. This fixes k and h , so the composition $F' := F \circ (\text{Id}_{\text{Im } \sigma} + F_T)$ has

$$(F'^\# q)^h = ((\text{Id}_{\text{Im } \sigma} + F_T)^\# F^\# q)^h = (F^\# q)^h \circ F_T = q^h.$$

Hence $F' \in \text{Aut}_{q^\circ}$, and $F'(k) = F(k)$ implies $P(F') = P(F)$. \square

Example 4.11. For the base $(\mathbb{Z} \oplus \mathbb{Z}/2^j, \langle \frac{1}{2^j} \rangle, (2^j, 0))$ of Example 4.7 let q° be the refinement with $q^{(2^{j-1}, 0)} = \langle \langle \frac{1}{2^{j+1}} \rangle \rangle$. The isomorphism F of the base in Example 4.7 does not preserve q° : if $j > 1$ then F alters the homogeneity defect of $q^{(2^{j-1}, 0)}$ and if $j = 1$ F alters the Arf invariant. However, if $j \geq 3$ then $F' : (x, y) \mapsto (x, x + (2^{j-1} + 1)y)$ is an isomorphism of q° with $\tilde{P}(F') = P(F) = 2^m \pmod{2^{j+1}}$.

The following examples illustrate that r , and hence $\text{Im } \tilde{P}$, can depend on b as well as (G, p) .

Example 4.12. Let $G = \mathbb{Z} \oplus (\mathbb{Z}/2^j)^2$ and $p = (2^j, 0, 0)$ (so $d_\pi = d_o = 2^j$). Choosing the torsion form $b_0 = \langle \frac{1}{2^j} \rangle \oplus \langle \frac{1}{2^j} \rangle$ on TG , using Example 4.7 shows that $r = 0$.

Let b_1 be the hyperbolic torsion form on TG with matrix

$$\begin{pmatrix} 0 & 2^{-j} \\ 2^{-j} & 0 \end{pmatrix}.$$

Since $d_\pi = d_o$, it follows that $r_1 = 0$ or 1 . But b_1 contains no split cyclic summands, and so by Lemma 4.4, we conclude that $r_1 = 1$.

The next example shows that r cannot be determined merely from the type of splitting of b (cyclics versus hyperbolic), but can depend on the isomorphism classes of split cyclic summands.

Example 4.13. Let $G = \mathbb{Z} \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/64 \oplus \mathbb{Z}/512$ with torsion form $\langle \frac{1}{8} \rangle \oplus \langle \frac{1}{64} \rangle \oplus \langle \frac{\epsilon}{512} \rangle$ ($\epsilon = \pm 1$), and $p = (64, 0, 8, 0)$. Then $d_\pi = 64$ and $d_o = 8$, so r is 0 or 1 . The 2-extremal exponents are 0 and 3 . If $F \in \text{Aut}_b$ then by (41)

$$P(F) = d_o \pmod{2d_o} \Leftrightarrow (\text{Id} - F)(1, 0, 0, 0) \text{ split } 512\text{-torsion} \Leftrightarrow (\text{Id} - F)(8, 0, 1, 0) \text{ split } 8\text{-torsion}.$$

Thus if $r = 0$ there must be some automorphism f of (T, b) such that $(\text{Id} - f)(0, 1, 0)$ plus a split 8-torsion element is divisible by 8 , i.e. $f(0, 1, 0) = (a, 8b + 1, 8c)$ with a odd. If $\epsilon = +1$ then the would-be image has norm $\frac{17}{64}$ for any a, b, c , so there can be no such f ; hence $r = 1$. On the other hand, if $\epsilon = -1$ we can define such an f by the matrix

$$\begin{pmatrix} 1 & -8 & 0 \\ 1 & 1 & 8 \\ 1 & 1 & 1 \end{pmatrix}.$$

Setting $F = \text{Id}_\mathbb{Z} + \rho + f$ with $\rho : \mathbb{Z} \rightarrow T, n \mapsto (0, 0, -n)$ makes $\text{Id} - F$ map $(8, 0, 1, 0) \in G$ to the split 8-torsion element $(1, 0, 0) \in T$ (and $(1, 0, 0, 0)$ to $(0, 0, -1)$), so $P(F) = d_o \pmod{2d_o}$, and $r = 0$.

Corollary 4.14. *Modulo the action of Aut_{q° , the number of possible Gauss refinements of (G, q°, p) is*

$$\text{Num} \left(\frac{2^r d_o}{8} \right),$$

and the number of possible mod 28 Gauss refinements is

$$\gcd \left(28, \text{Num} \left(\frac{2^r d_o}{8} \right) \right).$$

Remark 4.15. Notice that Corollary 4.14 combined with Theorems 1.2 and 1.3 gives the computation of the inertia group $I(M)$ for 2-connected M from Theorem 1.9.

4.4. The computation of reactivity. In this subsection we use Proposition 4.10 to prove lower bounds on the reactivity of every spin 7-manifold M . When M is 2-connected we also prove that this lower bound is sharp and so compute the reactivity of 2-connected M . Recall from Section 3.3 that if f is a self-almost diffeomorphism of M , then the mapping torus T_f is almost smooth, the spin characteristic class $p_{T_f} \in H^4(T_f)$ is well defined and so is the integer

$$p^2(f) = \langle p_{T_f}^2, [T_f] \rangle \in 8\mathbb{Z}.$$

The bridge between the algebraic arguments of §4.2-4.3 and the computation of reactivity of M , as defined in (5), is the following

Lemma 4.16. $\tilde{P}(f^*) = p^2(f) \bmod 2\tilde{d}_\pi$ for any self-almost diffeomorphism of M .

Proof. Let $f: M \cong M$ be an almost diffeomorphism and let W be a 3-connected spin coboundary for M . By definition we can use W to compute the generalised Eells-Kuiper invariant of M : $\mu_M = g_W \bmod \tilde{d}_\pi$. We can also use f to glue two copies of W together along M and form the almost smooth spin manifold $X := (-W) \cup_f W$. By Lemma 3.12, $p^2(f) = p_X^2$. From the definition of $\tilde{P}(f)$ in (43) and from the comparison of Gauss refinements in (24) we have

$$\tilde{P}(f^*) \equiv 8(g_W - (f^*)^\sharp g_W) \equiv p_X^2 - \sigma(X) \equiv p_X^2 \equiv p^2(f) \bmod 2\tilde{d}_\pi,$$

where $\sigma(X) = \sigma(W) - \sigma(W) = 0$ by Novikov additivity, and all equivalences are mod $2\tilde{d}_\pi$. \square

Corollary 4.17. For any closed spin 7-manifold M we have that:

- (i) $R_H(M)$ is divisible by $2\tilde{d}_\pi$;
- (ii) $R(M)$ is divisible by $\text{lcm}(8, 2^r d_o)$;
- (iii) $R_H^{\text{Diff}}(M)$ is divisible by $\text{lcm}(224, 2\tilde{d}_\pi)$;
- (iv) $R^{\text{Diff}}(M)$ is divisible by $\text{lcm}(224, 2^r d_o)$.

If M is 2-connected then equality holds in each case.

Proof. For part (i), Lemma 4.16 shows that $R_H(M) \mid 2\tilde{d}_\pi$, while Proposition 3.10 shows that $R_H(M) \mid 2\tilde{d}_\pi$ if M is 2-connected. For part (ii), let M have refinement (G, q°, p) . Proposition 4.10 computes the image $\text{Im } \tilde{P} = \text{lcm}(8, 2^r d_o)\mathbb{Z}/2\tilde{d}_\pi$, so Lemma 4.16 gives $\text{lcm}(8, 2^r d_o) \mid R(M)$. On the other hand, if M is 2-connected then Theorem 1.2 states that every automorphism of (G, q°, p) is realised by an almost diffeomorphism $f: M \cong M$, and so part (i) and Proposition 4.10 imply the equality. Parts (iii) and (iv) follow from parts (i) and (ii) and Lemma 3.9. \square

Proof of Theorem 1.9. The computation of $R(M) = \text{lcm}(8, 2^r d_o)$ is given in Corollary 4.17 (ii). Then $I(M) = \text{Num}(\frac{2^r d_o}{8})\Theta_7$ by Proposition 3.8 (i). By Remark 4.6, $r = 1$ if $TH^4(M)$ is of odd order. \square

5. EXAMPLES

Ever since Milnor's discovery of exotic 7-spheres [27], 2-connected 7-manifolds have provided interesting examples in topology and geometry. In this section we discuss a number of examples of 2-connected 7-manifolds. In Section 5.1 we consider the total spaces of 3-sphere bundles over S^4 and their connected sums. In Section 5.2 we mention some examples admitting interesting metrics. In Section 5.3 we give examples which are tangentially homotopy equivalent but not homeomorphic. Finally in Section 5.4 we present a refinement of Wilkens' list [38, Theorem 1] of the indecomposable generators for the monoid of almost diffeomorphism classes of 2-connected 7-manifolds.

5.1. 3-sphere bundles over S^4 and their connected sums. Following the notation of [9], let (n, p) be integers with same parity and let $M_{n,p} := S(\xi_{n,p})$ denote the total space of the 3-sphere bundle over S^4 which there corresponding vector bundle $\xi_{n,p}$ has Euler class $e(\xi_{n,p}) = n \in H^4(S^4)$ and spin characteristic class $\frac{p-1}{2}(\xi_{n,p}) = p \in H^4(S^4)$. By definition, we $M_{0,p} = M(\mathbb{Z}, p)$. Using (12) and (18) and recalling the notation of Example 2.5, we compute that for $n \neq 0$ there is a diffeomorphism

$$M_{n,p} \cong M\left(\left\langle\left\langle\frac{-1}{2n}\right\rangle\right\rangle_{-p}, \frac{p^2 - |n|}{8n}\right).$$

Example 5.1. The Milnor sphere, $\Sigma_M := M_{1,3}$, is homeomorphic to S^7 but not diffeomorphic to S^7 since $\mu(\Sigma_M) = 1 \neq 0 \bmod 28$: see [27] and [12].

In [8] the total spaces of 3-sphere bundles over S^4 were classified up to homotopy homeomorphism and diffeomorphism.

We now give an example which illustrates the subtleties of the inertia group. Building on Examples 4.7 and 4.8, Theorem 1.9 gives to the following

Example 5.2. The connected sums

$$M_0 := M_{-8,0} \# M_{0,8}, \quad M_1 := M_{-8,2} \# M_{0,8} \quad \text{and} \quad M_2 := M_{-8,4} \# M_{0,8},$$

have $r(M_i) = i$. In each case $d_\pi(M_i) = 8$, whereas $d_o(M_0) = 8$, $d_o(M_1) = 2$ and $d_o(M_2) = 4$. From Theorem 1.9 we have $I(M_0) \cong I(M_1) \cong \Theta_7$ and $I(M_2) \cong 2\Theta_7$.

Notice that when $r = 1$ the [39, Conjecture p.548] correctly predicts $I(M_1) = \Theta_7$. However when $r \neq 1$, [39, Conjecture p.548] incorrectly predicts that $I(M_0)$ is $2\Theta_7$ and that $I(M_2)$ is Θ_7 .

Example 5.3. While [39, Theorem 1] and Theorem 1.9 give $I(M(\mathbb{Z}^b, d)) = \text{Num}(\frac{d}{4}) \Theta_7$, the classical Eells-Kuiper invariant is not defined for $M(\mathbb{Z}^b, d)$ when $d_\pi = d \neq 0$. Using (18) we compute that

$$\mu(M(\mathbb{Z}^b, d) \# \Sigma) = [\mu(\Sigma)] \in \mathbb{Z}/\hat{d}_\pi \mathbb{Z}.$$

Hence for the Minor sphere Σ_M we have $\mu(M(\mathbb{Z}^b, 8) \# \Sigma_M) = 1 \in \mathbb{Z}/2\mathbb{Z}$, whereas $\mu(M(\mathbb{Z}^b, 8)) = 0$, and the generalised Eells-Kuiper invariant distinguishes the diffeomorphism types of $M(\mathbb{Z}^b, 8)$ and $M(\mathbb{Z}^b, 8) \# \Sigma_M$.

We can also deduce from Theorem 1.5 that, for example, $M(\mathbb{Z}^b, 8) \# \Sigma_M$ admits an orientation reversing diffeomorphism, whereas $M(\mathbb{Z}^b, 16) \# \Sigma_M$ does not.

Example 5.4. If N is a simply connected oriented 6-manifold with $\pi_2(N) \cong \mathbb{Z}$ and $S^1 \rightarrow M \rightarrow N$ is a principle S^1 bundle with primitive first Chern class, then M is 2-connected with a preferred orientation and hence spin structure. Conversely, by [19, Lemma 2.1], every free S^1 action on M is equivalent to such a principle bundle action. In [19, Theorem 1.3] Yi Jiang identifies the homeomorphism types and diffeomorphism types of all 2-connected M which admit free circle actions. In particular, by [19, Theorem 1.3] every such M is almost diffeomorphic to a connected sum $M_{bk, b(k+12m)} \#_{2r} M_{0,0}$ for $b \in \{1, 2\}$, $r \in \mathbb{Z}^{\geq 0}$ and $m, k \in \mathbb{Z}$.

5.2. Examples from geometry. There are a number of examples of 2-connected 7-manifolds admitting metrics with interesting geometric properties. Indeed by [7, Theorem B], every 2-connected 7-manifold admits a metric with positive Ricci curvature.

The smooth manifold underlying the Gromoll-Meyer sphere Σ_{GM} [16] is an exotic 7-sphere admitting a metric of non-negative sectional curvature. If $Sp(n)$ denotes the n -dimensional symplectic group of orthogonal $n \times n$ quaternionic matrices, then Σ_{GM} is a certain quotient of $Sp(2) \times Sp(1)$ by $Sp(1) \times Sp(1)$. By [16, Theorem 1], $\Sigma_{GM} \cong M_{-1,-5} \cong 3\Sigma_M$.

The smooth manifold underlying the Berger space B is a homogeneous space $B = SO(5)/SO(3)$ (where $SO(3) \hookrightarrow SO(5)$ by the adjoint representation) that admits a metric of positive sectional curvature. The Berger space is 2-connected with $H^4(B) \cong \mathbb{Z}/10$, and Goette, Kitchloo and Shankar [15, Corollary 2] proved there is a spin diffeomorphism

$$B \cong M_{10,8}.$$

More recently Grove, Verdiani and Ziller [17, Theorem A] constructed a metric of positive sectional curvature on a 2-connected manifold P_2 with $H^4(P_2) \cong \mathbb{Z}/2$. Applying [5, Theorem A], they deduced that there is an almost diffeomorphism $P_2 \cong S(TS^4)$, where $S(TS^4) = M_{2,0}$ is the unit tangent sphere bundle of S^4 . In [13] Goette proved that there is a diffeomorphism

$$P_2 \cong M_{2,2} \# (-\Sigma_M).$$

In [4] Corti, Haskins, the second author and Pacini constructed a very large class of examples of simply connected manifolds with G_2 holonomy metrics. Many of these examples are 2-connected with $H^4(M)$ torsion-free. For instance, [4, Table 3] gives 7 explicit ways to construct holonomy G_2 metrics on $M(\mathbb{Z}^{85}, 2)$. By Theorem 1.2, the underlying topological manifold admits a unique smooth structure. On the other hand, the same table lists 2 different ways to construct holonomy G_2 metrics with underlying topological manifold $M(\mathbb{Z}^{99}, 8)$. In this case $I(M) = 2\Theta(7)$, so M admits two inequivalent smooth structures, and to determine whether those two G_2 -manifolds are diffeomorphic one needs to evaluate their generalised Eells-Kuiper invariants.

5.3. Tangentially homotopy equivalent manifolds. Let N_0 and N_1 be closed smooth manifolds of the same dimension and let τ_{N_i} denote the stable tangent bundle of N_i . A tangential homotopy equivalence $f: N_0 \rightarrow N_1$ is a homotopy equivalence such that there is a stable vector bundle isomorphism $f^*\tau_{N_1} \cong \tau_{N_0}$. It is natural to ask under what conditions tangentially homotopy equivalent manifolds are necessarily homeomorphic, and this question was studied in detail by Madsen, Taylor and Williams in [26].

In [5, p. 144] it was proven the 2-connected manifolds give rise to examples of non-homeomorphic tangentially homotopy equivalent manifolds. We present a simplified version of the proof here, which starts with the following

Lemma 5.5. *Let M_0 and M_1 be 2-connected and let $f: M_0 \simeq M_1$ be a homotopy equivalence such that $(f^*)^\# p_{M_1} = p_{M_0}$. Then f is tangential.*

Proof. Let \widetilde{KO} denote reduced real K -theory. By [21] the spin characteristic class p defines an isomorphism $p: \widetilde{KO}(S^4) \cong H^4(S^4)$. The Atiyah-Hirzebruch spectral sequence for $\widetilde{KO}(M_0)$ then shows that p defines an isomorphism $p: \widetilde{KO}(M) \cong H^4(M)$. By definition, $p_{M_i} = p(\tau_{M_i})$ and so the assumption $(f^*)^\# p_{M_1} = p_{M_0}$ gives $f^*\tau_{M_1} \cong \tau_{M_0}$, which is to say that f is tangential. \square

Proposition 5.6 (cf. [5, p. 114]). *The manifolds $M_{8,1}$ and $M_{8,25}$ are tangentially homotopy equivalent but not homeomorphic.*

Proof. We first show that $M_{8,1}$ and $M_{8,25}$ are homotopy equivalent. Let $SG(4)$ denote the topological monoid of orientation preserving self-homotopy equivalences of S^3 . Elements of $SG(4)$ are clutching functions for spherical fibrations over S^4 with fibre S^3 . The natural map $i: SO(4) \in SG(4)$ induces a map on homotopy groups $i_*: \pi_3(SO(4)) \rightarrow \pi_3(SG(4))$. In [8, p. 337] it is shown that $\pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}/12$ and that

$$i_*(\xi_{n,p}) = \left(n, \left[\frac{p-n}{2} \right] \right) \in \mathbb{Z} \oplus \mathbb{Z}/12,$$

where $\left[\frac{p-n}{2} \right] \in \mathbb{Z}/12$ is the mod 12 reduction of $\left[\frac{p-n}{2} \right]$. Hence the bundles $\xi_{8,1}$ and $\xi_{8,25}$ are fibre homotopy equivalent and so there is a fibre homotopy equivalence $f: M_{8,1} \simeq M_{8,25}$ between their sphere bundles.

To see that f is tangential, let $M_0 = M_{8,1}$, $M_1 = M_{8,25}$, let $x \in H^4(S^4)$ be a fixed generator and let $x_i \in H^4(M_i)$ be the pull back of x to under the bundle map $M_i \rightarrow S^4$. By definition $p(M_i) = x_i$. Since f is a fibre homotopy equivalence $(f^*)^\# x_1 = x_0$. Hence $(f^*)^\# p_{M_1} = p_{M_0}$ and so f is tangential by Lemma 5.5.

To see that $M_0 := M_{8,1}$ and $M_1 := M_{8,25}$ are not homeomorphic, we use Proposition 2.10 to compute that $A(q_{M_0}) = -7/64 \bmod \mathbb{Z}$ but $A(q_{M_1}) = -23/64 \bmod \mathbb{Z}$. By Theorem 1.2, the refinement q is a homeomorphism invariant, and hence $M_{8,1}$ and $M_{8,25}$ are not homeomorphic. \square

Remark 5.7. Proposition 5.6 contradicts [26, Theorem C and Theorem 5.10] where it is stated, amongst other things, that all tangentially homotopy equivalent 2-connected 7-manifolds are homeomorphic. The source of the mistake in the arguments of [26] can be found in [26, Theorem 3.12] which is not correct. It is claimed that a certain cohomology class

$$f^*\pi^*(l_n) \in H^{4n}(S^2\Omega^2(SG[3, \infty]); \mathbb{Z}_{(2)})$$

vanishes. Here $SG[3, \infty]$ is the 2-connected cover of SG , $S^2\Omega^2$ denotes the double suspension of the double loop space, the coefficient group $\mathbb{Z}_{(2)}$ is the integers localised at 2 and we shall not define the maps f or π . However the argument given for the proof of [26, Theorem 3.12] only shows that $f^*\pi^*(l_n) = 2x$ for some $x \in H^{4n}(S^2\Omega^2(SG[3, \infty]); \mathbb{Z}_{(2)})$ and not that $f^*\pi^*(l_n) = 0$. To the best of our knowledge, this is the only flaw in the arguments of [26].

5.4. Generators for the monoid of 2-connected 7-manifolds. The connected sum operation gives the set of spin diffeomorphism classes of 2-connected 7-manifolds the structure of a commutative monoid with unit S^7 . Owing to the existence of homotopy 7-spheres, every M has non-trivial connected sum splittings

$$M \cong (M \# \Sigma) \# (-\Sigma)$$

for each $\Sigma \in \Theta_7$. Hence we call M *topologically decomposable* if there is a diffeomorphism

$$M \cong M_0 \# M_1$$

where neither M_0 nor M_1 is a homotopy sphere, and we call M *topologically indecomposable* if M is not decomposable.

By Theorem 1.5, every connected sum splitting of M gives rise to an orthogonal splitting of the refinement of M , and by Theorem 1.3 every orthogonal splitting of the refinement of M is realised by a connected sum splitting of M . Hence we call a refinement (G, q°, p) or a base (G, b, p) decomposable if it can be written as a non-trivial orthogonal sum and indecomposable otherwise. It is clear from the definitions that a refinement is indecomposable if and only if its base is indecomposable. Moreover, the indecomposable bases are of the form $(\mathbb{Z}, 0, p)$ and (T, b, p) where b is an indecomposable torsion form; *i.e.* b cannot be written as a non-trivial orthogonal sum. In this case we also call q indecomposable. A list of all isomorphism classes of indecomposable torsion forms was given by Wall [35, Theorem 4] and torsion forms were then classified by Kawauchi and Kojima [20, Theorem 4.1]. We do not go into details but note that if (T, b, p) is indecomposable then $T \cong \mathbb{Z}/p^k$ for a prime p or $T \cong (\mathbb{Z}/2^k)^2$. Summarising the above discussion we have the following refinement of a theorem of Wilkens.

Theorem 5.8 (cf. [38, Theorem 1]). *Every 2-connected M is diffeomorphic to a connected sum of topologically indecomposable manifolds M_i :*

$$M \cong \#_{i=1}^n M_i.$$

Moreover, M is topologically indecomposable if and only if it is almost diffeomorphic to a manifold of one of the following forms

$$S^7, \quad M(\mathbb{Z}, d), \quad M(q, s),$$

where in the final case, q is a prime refinement and hence $H^4(M(q, s)) \cong \mathbb{Z}/p^k$ for p a prime or $H^4(M(q, s)) \cong (\mathbb{Z}/2^k)^2$. \square

Even up to almost diffeomorphism, the splitting $M \cong \#_{i=1}^n M_i$ of Theorem 5.8 is in general far from being unique. For example the manifolds $M_{4,0}$ and $M_{4,2}$ have non-isomorphic bases $(\mathbb{Z}/4, \langle \frac{-1}{4} \rangle, 0)$ and $(\mathbb{Z}/4, \langle \frac{-1}{4} \rangle, 2)$ respectively, but $M_{4,0} \# M_{0,2}$ and $M_{4,2} \# M_{0,2}$ are diffeomorphic. Even when H^4 is torsion, there are many examples of torsion forms where $b_0 \oplus b_2 \cong b_1 \oplus b_3$ but b_0 is not isomorphic to b_1 or b_3 and the same holds for b_2 : see for example [20, §3]. This leads to non-uniqueness of connected sum splittings for manifolds with linking form isomorphic to $b_0 \oplus b_2$.

6. MAPPING CLASS GROUPS AND INERTIA

In this section we point out some implications of our classification results for mapping class groups of 2-connected M . Throughout this section M will be 2-connected.

Recall that $\text{Diff}(M)$ and $\text{ADiff}(M)$ denote the groups of spin diffeomorphisms of M and spin almost diffeomorphisms of M respectively. Recall also that $\text{Diff}_H(M)$ and $\text{ADiff}_H(M)$ denote subgroups of the above automorphism groups whose elements induce the identity on $H^*(M)$. For brevity, let $\text{Aut}_\mu(H^4(M))$ denote the group of automorphisms of the mod 28 distillation $(H^4(M), q_M^\circ, \mu_M, p_M)$ and let $\text{Aut}_{q^\circ}(H^4(M))$ denote the group of automorphisms of the classifying triple $(H^4(M), q_M^\circ, p_M)$. As an immediate consequence of Theorem 1.3 we obtain

Proposition 6.1. *For each 2-connected M , there is a short exact sequence*

$$0 \rightarrow \tilde{\pi}_0 \text{Diff}_H(M) \rightarrow \tilde{\pi}_0 \text{Diff}(M) \rightarrow \text{Aut}_\mu(H^4(M)) \rightarrow 0. \quad \square$$

Remark 6.2. The exact sequence of Proposition 6.1 serves as a starting point to study the mapping class groups $\tilde{\pi}_0 \text{Diff}(M)$. The determination of $\tilde{\pi}_0 \text{Diff}_H(M)$ and the extension in the sequence of Proposition 6.1 lie outside the scope of this paper. For example, we do not currently know if $\tilde{\pi}_0 \text{Diff}_H(M)$ is abelian in general.

We now consider $\tilde{\pi}_0\text{ADiff}(M)$, the mapping class group of almost diffeomorphisms of M . We recall the homomorphism $\tilde{P}: \text{Aut}_{q^\circ}(H^4(M)) \rightarrow \text{lcm}(8, 2^r d_o)\mathbb{Z}/2\hat{d}_\pi\mathbb{Z}$ from Section 4.3: see (43) and Proposition 4.10. Recalling that $\hat{d}_\pi = \gcd(\frac{d_o}{4}, 28)$, we then define

$$\hat{P}: \text{Aut}_{q^\circ}(H^4(M)) \rightarrow \text{Num}(2^{r-3}d_o)\mathbb{Z}/\hat{d}_\pi\mathbb{Z}, \quad F \mapsto \frac{\tilde{P}(F)}{8} \mod \hat{d}_\pi,$$

to be the mod \hat{d}_π reduction of \tilde{P} divided by 8. Notice that by Theorem 1.9 and Remark 3.11 we have $I(M)/I_H(M) \cong \text{Num}(2^{r-3}d_o)\mathbb{Z}/\hat{d}_\pi\mathbb{Z}$ and so we can equally regard \hat{P} as a homomorphism $\hat{P}: \text{Aut}_{q^\circ}(H^4(M)) \rightarrow I(M)/I_H(M)$.

Theorem 6.3. *For each 2-connected M there is a commutative diagram of group homomorphisms with short exact sequences for rows and with exact columns:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{\pi}_0\text{Diff}_H(M) & \longrightarrow & \tilde{\pi}_0\text{Diff}(M) & \longrightarrow & \text{Aut}_\mu(H^4(M)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \tilde{\pi}_0\text{ADiff}_H(M) & \longrightarrow & \tilde{\pi}_0\text{ADiff}(M) & \longrightarrow & \text{Aut}_{q^\circ}(H^4(M)) & \longrightarrow & 0 \\ & & \downarrow \partial_H & & \downarrow \partial & & \downarrow \hat{P} & & \\ 0 & \longrightarrow & I_H(M) & \longrightarrow & I(M) & \longrightarrow & I(M)/I_H(M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

In particular $F \in \text{Aut}_{q^\circ}(H^4(M))$ is realised by a diffeomorphism of M if and only if $\hat{P}(F) = 0$.

Proof. The top row is the exact sequence of Proposition 6.1. The exactness of the second row follows from Theorem 1.2. The first two columns are exact by the discussion at the beginning of Section 3.3, and in particular (34), and the third column is exact by the definition of \hat{P} . The only part of the commutativity of the diagram which needs comment is the bottom right hand square where the commutativity follows from Lemma 3.7 (ii) and Lemma 4.16. The final statement follows from the exactness of final row and the top column. \square

We shall call an almost diffeomorphisms *exotic* if it is not pseudo-isotopic to a diffeomorphism. A feature of the diagram in Proposition 6.3 is that when $I(M)/I_H(M) \neq 0$, M admits exotic almost diffeomorphisms which are detected by their action on $H^4(M)$. Specifically, if $f: M \cong M$ is an almost diffeomorphism, then $\hat{P}(f^*)$ is the obstruction to $f^*: H^4(M) \cong H^4(M)$ being induced by *any* diffeomorphism of M . Since \hat{P} is onto, it is enough to find cases where $I(M)/I_H(M)$ is non-zero to show that \hat{P} is non-zero.

Proposition 6.4. *Any pair of subgroups $I_0 \subseteq I_1 \subseteq \Theta_7$ can arise as the pair of inertia groups $(I_0, I_1) = (I_H(M), I(M))$ for some 2-connected M .*

Proof. There are three pairs of subgroups (T_0, T_1) in $\mathbb{Z}/7$ and six pairs of subgroups (T_0, T_1) in $\mathbb{Z}/4$, leaving 18 cases to realise. By Theorem 1.9 and Remark 3.11, $I(M)$ and $I_H(M)$ depend only on the base (G, b, p) . We list manifolds, their bases and the pairs of inertia groups they realise in the following table, where it is helpful to note that $102 = 7 \times 16$:

M	(G, b, p)	(I_0, I_1)	I_1/I_0
S^7	$(0, 0, 0)$	$(0, 0)$	0
$M_{0,56}$	$(\mathbb{Z}, 0, 56)$	$(\mathbb{Z}/2, \mathbb{Z}/2)$	0
$M_{0,28}$	$(\mathbb{Z}, 0, 28)$	$(\mathbb{Z}/4, \mathbb{Z}/4)$	0
$M_{0,16}$	$(\mathbb{Z}, 0, 16)$	$(\mathbb{Z}/7, \mathbb{Z}/7)$	0
$M_{0,8}$	$(\mathbb{Z}, 0, 8)$	$(\mathbb{Z}/14, \mathbb{Z}/14)$	0
$M_{0,4}$	$(\mathbb{Z}, 0, 4)$	$(\mathbb{Z}/28, \mathbb{Z}/28)$	0
$M_{-16,0} \# M_{0,102}$	$(\mathbb{Z}/16 \oplus \mathbb{Z}, \langle 1/16 \rangle, (0, 102))$	$(0, \mathbb{Z}/2)$	$\mathbb{Z}/2$
$M_{-8,0} \# M_{0,56}$	$(\mathbb{Z}/8 \oplus \mathbb{Z}, \langle 1/8 \rangle, (0, 56))$	$(\mathbb{Z}/2, \mathbb{Z}/4)$	$\mathbb{Z}/2$
$M_{-16,2} \# M_{0,102}$	$(\mathbb{Z} \oplus \mathbb{Z}/16, \langle 1/16 \rangle, (102, 2))$	$(0, \mathbb{Z}/4)$	$\mathbb{Z}/4$
$M_{-16,0} \# M_{0,16}$	$(\mathbb{Z}/16 \oplus \mathbb{Z}, \langle 1/16 \rangle, (0, 16))$	$(\mathbb{Z}/7, \mathbb{Z}/14)$	$\mathbb{Z}/2$
$M_{-8,0} \# M_{0,8}$	$(\mathbb{Z}/8 \oplus \mathbb{Z}, \langle 1/8 \rangle, (0, 8))$	$(\mathbb{Z}/14, \mathbb{Z}/28)$	$\mathbb{Z}/2$
$M_{-16,2} \# M_{0,16}$	$(\mathbb{Z}/16 \oplus \mathbb{Z}, \langle 1/16 \rangle, (2, 16))$	$(\mathbb{Z}/7, \mathbb{Z}/28)$	$\mathbb{Z}/4$
$M_{-7,1} \# M_{0,102}$	$(\mathbb{Z}/7 \oplus \mathbb{Z}, \langle 1/7 \rangle, (1, 102))$	$(0, \mathbb{Z}/7)$	$\mathbb{Z}/7$
$M_{-7,1} \# M_{0,56}$	$(\mathbb{Z}/7 \oplus \mathbb{Z}, \langle 1/7 \rangle, (1, 56))$	$(\mathbb{Z}/2, \mathbb{Z}/14)$	$\mathbb{Z}/7$
$M_{-0,14} \# M_{0,14}$	$(\mathbb{Z}/7 \oplus \mathbb{Z}, \langle 1/7 \rangle, (1, 14))$	$(\mathbb{Z}/4, \mathbb{Z}/28)$	$\mathbb{Z}/7$
$M_{-7,1} \# M_{102,16}$	$(\mathbb{Z}/102 \oplus \mathbb{Z}, \langle 1/102 \rangle, (16, 102))$	$(0, \mathbb{Z}/14)$	$\mathbb{Z}/14$
$M_{-56,0} \# M_{0,56}$	$(\mathbb{Z}/56 \oplus \mathbb{Z}, \langle 1/56 \rangle, (8, 56))$	$(\mathbb{Z}/2, \mathbb{Z}/28)$	$\mathbb{Z}/14$
$M_{-102,2} \# M_{0,102}$	$(\mathbb{Z}/102 \oplus \mathbb{Z}, \langle 1/102 \rangle, (2, 102))$	$(0, \mathbb{Z}/28)$	$\mathbb{Z}/28$

□

Theorem 1.12 follows immediately from Theorem 6.3 and Proposition 6.4. We now give an explicit example.

Example 6.5. Let $M = M_{-102,2} \# M_{0,102}$ so that $H^4(M) = \mathbb{Z}/102 \oplus \mathbb{Z}$ and consider the automorphism of $(H^4(M), q_M^\circ, p_M)$ defined by

$$F = \begin{pmatrix} 1 & [1] \\ 0 & 1 \end{pmatrix} : \mathbb{Z}/102 \oplus \mathbb{Z} \cong \mathbb{Z}/102 \oplus \mathbb{Z}.$$

In this case $\hat{d}_\pi = 28$ and from the proof of Proposition 6.4, we see that M admits an almost diffeomorphism $f: M \cong M$ with $f^* = F$, and $\hat{P}(f) = 1 \in \mathbb{Z}/28\mathbb{Z}$. By Theorem 6.3, F^n is realised by a diffeomorphism of M if and only if $n \equiv 0 \pmod{28}$.

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Diarmuid Crowley

Max Planck Institute for Mathematics
 Vivatsgasse 7, 53111 Bonn, Germany
 diarmuidc23@gmail.com

Johannes Nordström

Department of Mathematical Sciences
 University of Bath
 Claverton Down, Bath BA2 7AY, United Kingdom
 j.nordstrom@bath.ac.uk